Mathematics as a Chain of Proven Steps

The 63 fields of mathematics, from the first proofs to 2026

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October 2026

Contents

Preface 1

The 63 fields 2

Numbers and foundations 4

Number theory 9

Geometry 15

Algebra 20

Analysis 26

Topology 32

Probability and statistics 37

Combinatorics and graph theory 42

Differential equations and dynamical systems 47

Logic and computation 52

Numerical mathematics 56

Unifying structures 61

Order and lattices 65

Universal algebra 71

Linear algebra 76

Associative rings and algebras 83

Non-associative algebras 89

Algebraic geometry 94

Category theory and homological algebra 103

K-theory 110

Lie groups 117

Complex analysis 125

Potential theory 131

Several complex variables **136**

Special functions **142**

Difference and functional equations **149**

Series and summability **154**

Approximation theory **159**

Abstract harmonic analysis **164**

Integral transforms **169**

Integral equations **173**

Operator theory **177**

Calculus of variations and optimisation **181**

Convex and discrete geometry **186**

Global analysis **191**

Computer science **196**

Mechanics **202**

Elasticity and solids **207**

Fluid mechanics **212**

Optics and electromagnetism **218**

Thermodynamics **224**

Quantum theory **228**

Statistical mechanics **233**

Relativity **239**

Astronomy **246**

Geophysics **254**

Operations research **261**

Game theory and economics **266**

Mathematical biology **273**

Control theory **279**

Information theory **286**

Mathematics education **293**

General **298**

History of mathematics **309**

Where the chains meet **314**

Sources and method **324**

Preface

This book tells the history of mathematics as a set of chains, one for each of its 63 fields. Each link in a chain is a result and the proof that settled it. Each step from one link to the next says what the later proof took from the earlier one.

Most histories of mathematics follow people or periods. This one follows proofs. The question it asks at every step is the same: what did this result rest on, and what came to rest on it? Read that way, a field shows its skeleton. Some results carry long chains behind them; others arrived as independent discoveries, with no line back to anything before.

How to read a chain

Each chapter opens with the main lines of its field and then tells them in order of time. For every link you will find who proved it, when, what was proved, and the idea of the proof in a sentence or two.

- **A step from one result to the next** means that the later proof or definition uses the earlier result.
- **Independent results** are named as such: two proofs found in parallel, a rediscovery, or a later result that does not depend on an earlier one that looks related.
- **Other kinds of step** are named too: a conjecture, a new definition, a counterexample, or a gap that a later proof repaired.
- **Quick checks** in brackets are numbers worked out again: a date difference, a formula, a bound or a numerical example.
- **Notes** at the end of each chapter record where sources disagree on a date, an attribution or a dependency.
- **Sources** at the end of each chapter are numbered and cited in the text as [1], [2] and so on.

Several chapters carry a diagram of their chain. A solid line means the proof uses the earlier result; a dashed line marks an influence, a reproof, or a step that is less firmly documented.

How it was checked

Every link rests on at least one opened source of standing: a scholarly history, an encyclopaedia of mathematics, the original paper, or a university text. Wikipedia was never accepted as the only source. Every number in a quick check was recomputed in Python. Where a source could not be found for a claim, the claim is either worded with care (“according to …”) or left out. The last chapter sets out the method in full.

The chains end in the present. The most recent results, from 2024 to 2026, are marked as preprints where no journal version had appeared when they were checked.

The 63 fields

The fields are the 63 top-level areas of the Mathematics Subject Classification (MSC2020), kept by the American Mathematical Society and zbMATH (msc2020.org). Each has a chapter of its own or shares one. Two chapters, Numbers and foundations and Unifying structures, cut across several fields.

MSC	Field	Chapter
00	General: philosophy, recreational mathematics, modelling	General
01	History and biography	History of mathematics
03	Mathematical logic and foundations	Logic and computation; Numbers and foundations
05	Combinatorics	Combinatorics and graph theory
06	Order, lattices, ordered algebraic structures	Order and lattices
08	General algebraic systems	Universal algebra
11	Number theory	Number theory
12	Field theory and polynomials	Algebra
13	Commutative algebra	Algebra
14	Algebraic geometry	Algebraic geometry
15	Linear and multilinear algebra; matrix theory	Linear algebra
16	Associative rings and algebras	Associative rings and algebras
17	Non-associative rings and algebras	Non-associative algebras
18	Category theory; homological algebra	Category theory and homological algebra
19	K-theory	K-theory
20	Group theory and generalizations	Algebra
22	Topological groups, Lie groups	Lie groups
26	Real functions	Analysis
28	Measure and integration	Analysis
30	Functions of a complex variable	Complex analysis
31	Potential theory	Potential theory
32	Several complex variables and analytic spaces	Several complex variables
33	Special functions	Special functions
34	Ordinary differential equations	Differential equations and dynamical systems
35	Partial differential equations	Differential equations and dynamical systems
37	Dynamical systems and ergodic theory	Differential equations and dynamical systems
39	Difference and functional equations	Difference and functional equations

MSC	Field	Chapter
40	Sequences, series, summability	Series and summability
41	Approximations and expansions	Approximation theory
42	Harmonic analysis on Euclidean spaces	Analysis
43	Abstract harmonic analysis	Abstract harmonic analysis
44	Integral transforms, operational calculus	Integral transforms
45	Integral equations	Integral equations
46	Functional analysis	Analysis
47	Operator theory	Operator theory
49	Calculus of variations and optimal control; optimization	Calculus of variations and optimisation
51	Geometry	Geometry
52	Convex and discrete geometry	Convex and discrete geometry
53	Differential geometry	Geometry
54	General topology	Topology
55	Algebraic topology	Topology
57	Manifolds and cell complexes	Topology
58	Global analysis, analysis on manifolds	Global analysis
60	Probability theory and stochastic processes	Probability and statistics
62	Statistics	Probability and statistics
65	Numerical analysis	Numerical mathematics
68	Computer science	Computer science
70	Mechanics of particles and systems	Mechanics
74	Mechanics of deformable solids	Elasticity and solids
76	Fluid mechanics	Fluid mechanics
78	Optics, electromagnetic theory	Optics and electromagnetism
80	Classical thermodynamics, heat transfer	Thermodynamics
81	Quantum theory	Quantum theory
82	Statistical mechanics, structure of matter	Statistical mechanics
83	Relativity and gravitational theory	Relativity
85	Astronomy and astrophysics	Astronomy
86	Geophysics	Geophysics
90	Operations research, mathematical programming	Operations research
91	Game theory, economics, social and behavioral sciences	Game theory and economics
92	Biology and other natural sciences	Mathematical biology
93	Systems theory; control	Control theory
94	Information and communication, circuits	Information theory
97	Mathematics education	Mathematics education

Numbers and foundations

This chain has two halves. From tallies to Stevin it is a chain of repairs more than of proofs: each step fixes a gap in the number system before it. From 1872 on it is a chain of proofs in the strict sense. The reals are built from the rationals, Cantor's diagonal leads to Russell's paradox, and the axioms that block the paradox are the base on which Gödel and Cohen prove that the continuum hypothesis cannot be decided.

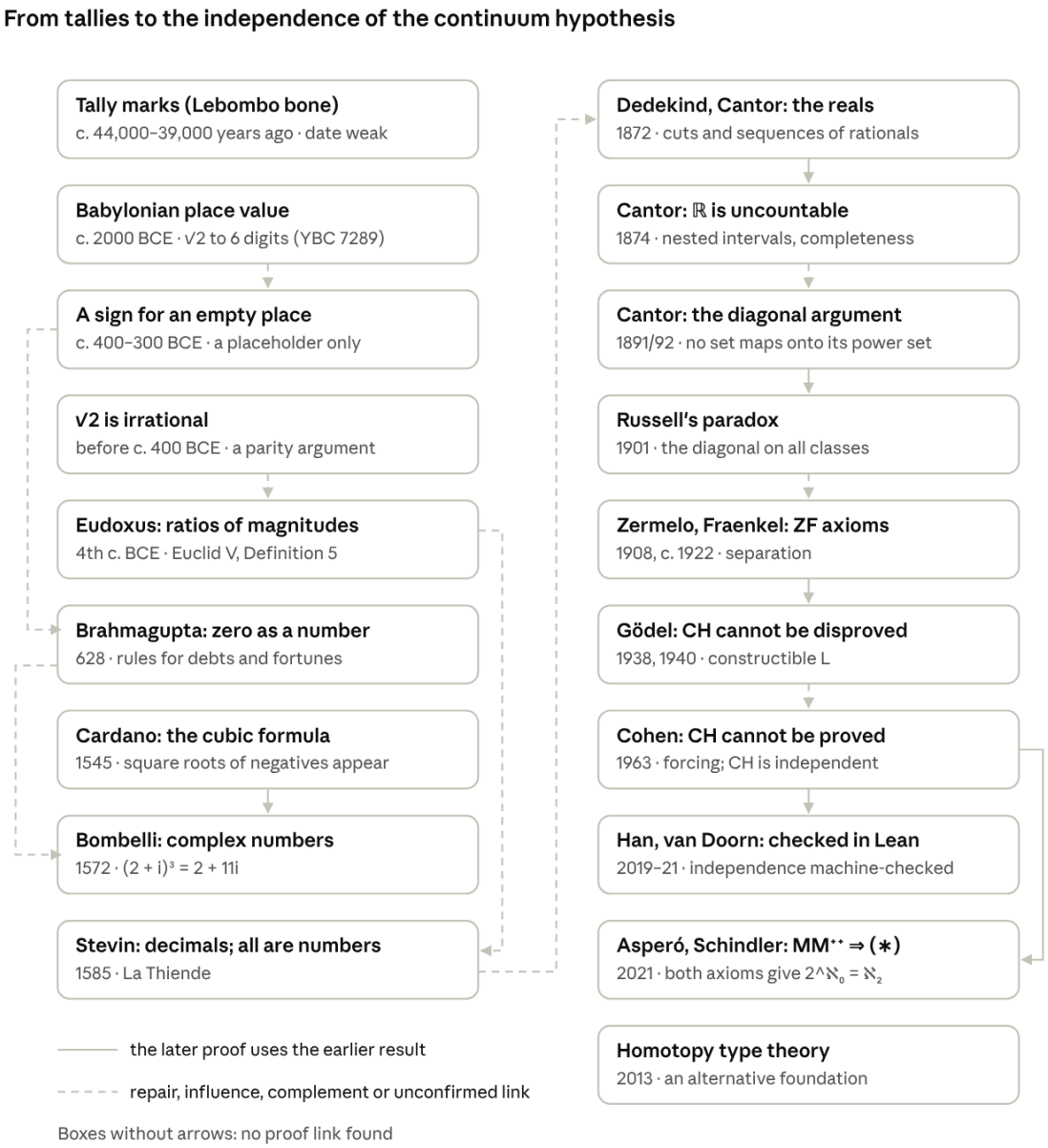


Figure 1: numbers and foundations · 19 links from tallies to Lean

From tally marks to place value and zero

The oldest link is not a proof at all. Notched bones from the Lebombo Mountains and Border Cave in southern Africa, some 44,000 to 39,000 years old, carry cuts that were used "for notational purposes" [3, 5, 6]. Britannica describes the earliest numerals in the same way, as "notches in a stick" [1]. Reading the notches as a one-to-one match between marks and things counted is an interpretation, not something the bones show, and the bone's own date differs between sources (note 1) [2, 4, 6].

By about 2000 BCE the Babylonians wrote numbers in place value: a digit's value depends on its position, as a multiple of 60^k [7, 8]. The tablet YBC 7289 gives √2 as 1;24,51,10 in base 60 [9, 10]. A quick check: $1 + 24/60 + 51/60^2 + 10/60^3 = 1.4142130$, which is off from √2 by only 6.0×10^{-7}. The sources draw no line from the tally marks to this system, so the two are best seen as separate developments.

Place value had a gap. For "over 1000 years" the Babylonians had no sign for an empty position [11]. Around 400 BCE, or by 300 BCE at the latest, they began to mark an empty place with two wedges (note 4) [11, 12]. The repair matters: in base 60, "1 _ 1" means 3601, while "1 1" means 61.

The Greek crisis: √2 and Eudoxus

Before about 400 BCE Greek mathematicians proved that √2 is not a ratio of whole numbers [13, 14]. The proof is short. If $\sqrt{2} = a/b$ in lowest terms, then $a^2 = 2b^2$, so a is even; writing $a = 2c$ gives $b^2 = 2c^2$, so b is even too, against the assumption. This refuted the belief that any two lengths stand in a ratio of whole numbers. MacTutor notes that the Babylonians worked with approximations to √2 long before the Greeks did, but presents this as a separate development; no source shows that the Greek proof grew from the Babylonian work [15].

In the 4th century BCE Eudoxus repaired the gap that √2 had opened (note 1). His definition, preserved in Euclid's Book V, avoids whole-number ratios altogether: a:b = c:d when, for all whole numbers m and n, ma > nb exactly when mc > nd [16, 17]. A quick check on the grid m, n = 1 … 199, with 50-digit decimals: $1:\sqrt{2} = 2:2\sqrt{2}$ holds for every pair, while $1:\sqrt{2} = 100:141$ fails, first at m = 24, n = 17. There $24 > 17\sqrt{2} \approx 24.04$ is false, but 2400 > 2397 is true.

Zero, negatives and the square roots of negatives

In 628 Brahmagupta treated zero as a number in its own right, defined as a − a [18, 19]. He gave rules for adding and multiplying debts and fortunes, our negative and positive numbers. He also set 0/0 = 0, which is wrong. MacTutor places both the place-value zero and zero as a number in Indian mathematics by about 650, but does not say that Brahmagupta's rules were derived from the place-value sign [11].

In 1545 Cardano published a formula for the cubic, found by writing the unknown as x = u + v [20, 22]. When the cubic has three real roots, the formula passes through square roots of negative numbers. Cardano did not know what to do with them [21].

In 1572 Bombelli gave rules for computing with $\pm\sqrt{-1}$, and these rules recover the real roots [21, 22, 23]. A quick check: for $x^3 = 15x + 4$, Cardano's formula needs the cube roots of 2 ± 11i. Since $(2 + i)^3 = 2 + 11i$, the two cube roots add to (2 + i) + (2

$-\,i) = 4$, and indeed $4^3 = 15 \cdot 4 + 4$. Bombelli's work rests on Cardano's formula. His rules of signs resemble Brahmagupta's, but no source shows that they were passed on from India to Italy.

In 1585 Simon Stevin introduced decimal fractions and argued that irrationals and negatives are numbers like any other [24, 26]. In this way he turned the Greeks' continuous magnitudes into numbers. MacTutor links Stevin's view to Euclid's Book X, where two magnitudes are incommensurable when the Euclidean algorithm does not end; no source opened names Eudoxus as his model [15, 25].

The real numbers built and counted

In 1872 Dedekind and Cantor each built the real numbers $\mathbb{R}$ from the rationals $\mathbb{Q}$ alone: Dedekind by cuts, Cantor by classes of Cauchy sequences [17, 27]. The two constructions give isomorphic systems. A quick check: bisecting the cut for $\sqrt{2}$, starting from the interval $[1, 2]$ and working in exact rationals, leaves a width of $2^{-40} \approx 9.1 \times 10^{-13}$ after 40 steps. According to Katz and Katz, Stevin's belief that one can compute with all reals had to wait for Dedekind's work for its rigorous justification; they do not say that Stevin motivated it [25]. Whether Dedekind's cut is Eudoxus's definition made exact is disputed (note 3).

In 1874 Cantor proved that $\mathbb{R}$ is uncountable [28, 29]. Given any sequence of reals, he built nested intervals, each avoiding the next term of the sequence; their common point is a real number not in the list. The proof rests on the completeness established in 1872. It also used Dedekind's proof that the algebraic numbers can be counted.

In 1891 (note 2) Cantor gave the diagonal argument: no map from a set S to its power set P(S) is onto [28, 30]. For any map f, the set of x with $x \notin f(x)$ is missed. This reproves the 1874 result by a general method. A quick check: of the 512 maps from $\{0, 1, 2\}$ to the eight subsets of $\{0, 1, 2\}$, none is onto, and in each the diagonal set is missing from the image.

Paradox, axioms and the continuum hypothesis

In 1901 Russell applied Cantor's diagonal to the class of all classes. The class $R = \{x : x \notin x\}$ contains itself exactly when it does not, so naive comprehension is inconsistent [33, 34]. Frege's Basic Law V falls with it.

Zermelo repaired the paradox in 1908 with the axiom of Separation: sets may only be cut out of sets that already exist [33, 35]. Around 1922 Fraenkel and Skolem added Replacement, giving the system now called ZF, or ZFC with the axiom of choice [36].

In 1938 Gödel proved that if ZF is consistent, so is ZFC with the generalised continuum hypothesis: $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$ [36, 37, 38]. He used the constructible universe L, built in stages from ZF as the base theory. The announcement appeared in PNAS 24(12) in December 1938, with the monograph following in 1940 [39].

In 1963 Paul Cohen proved the other half, $\mathrm{Con}(\mathrm{ZF}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH})$, by his new method of forcing (note 5) [40, 41]. Together with Gödel's result this shows that the continuum hypothesis is independent of ZFC. A quick check: Hilbert posed it as his first problem in 1900, so the answer came 63 years later. Cohen's result complements Gödel's; according to MacTutor he came to forcing by reading Gödel's monograph [40].

Between 2019 and 2021 Jesse Han and Floris van Doorn checked the independence of the continuum hypothesis by machine, in the Lean proof assistant, using Boolean-valued models [42, 43]. Their work rests on Cohen's forcing.

In 2021 Asperó and Schindler proved that the forcing axiom MM^{++} implies Woodin's axiom (∗) [44, 45]. Both axioms give 2^ℵ₀ = ℵ₂, so the two main candidate answers to the size of the continuum agree. The result belongs to the forcing tradition, since MM^{++} is itself a forcing axiom [36].

Homotopy type theory, set out in 2013, offers univalent foundations, in which equivalent types are equal [46, 47]. It has no arrow back to set theory: it is an alternative foundation, not a theorem proved from the earlier axioms.

Notes

1. Eudoxus is dated 408–355 BCE by MacTutor [16] and c. 395/390–342/337 BCE by Britannica; the conflict stands. The Lebombo bone's own date also stays open: 43,000–42,000 years in Wikipedia [2] and 39,000 years in the Heritage Portal [6]. The Heritage Portal and Wits report notational use at Border Cave c. 44,000 years ago [5, 6], while the d'Errico et al. PNAS 2012 abstract ties its c. 44,000-year date to "modern hunter–gatherer adaptation" in general [3, 4].
2. The diagonal argument is in Jahresbericht DMV 1, pp. 75–78, the volume for 1890–91 [30, 31, 32]. SEP gives 1892 [28]; that this is the printing year is not confirmed, and the volume itself was not opened.
3. Whether Dedekind's cut is Eudoxus's definition made exact is disputed. Heath, and Dedekind himself, say it is [16]. Gardies denies any influence, and SEP names Gauss and Dirichlet as Dedekind's models [17].
4. The two-wedge placeholder is dated c. 400 BC by MacTutor [11] and by 300 BC by Wikipedia [12].
5. Cohen's date is 1963. The 1961 in SEP's Gödel entry [41] is a slip; MacTutor gives the 1963 papers and lectures [40]. Gödel's 1938 date is fixed by the PNAS issue of December 1938 [39].

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Number theory

Number theory has some of the longest proof chains in mathematics. Euler's product formula of 1737 is the hinge: Dirichlet and Riemann start from it, and through Riemann so do the prime number theorem and the zero-density bound of 2024. A second line runs from a conjecture of the 1950s to Fermat's Last Theorem. A third runs from sieves to bounded gaps between primes, and borrows Deligne's bounds from algebraic geometry on the way.

Outside these chains, Gerd Faltings won the 2026 Abel Prize, announced on 19 March 2026, "for introducing powerful tools in arithmetic geometry and resolving long-standing diophantine conjectures of Mordell and Lang" [1, 2, 3]. He proved Mordell's 1922 conjecture in 1983.

From Euclid to Euler

Around 300 BCE Euclid proved that there are infinitely many primes [4, 5, 7]. Take any finite list of primes and let m be their least common multiple. Then m + 1 leaves remainder 1 on division by each of them, so its prime factors are not on the list. A quick check shows why the argument needs care: $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \times 509$, so m + 1 need not itself be prime. The proof rests on Elements VII.31, that every composite number has a prime factor, which both Joyce and the Stony Brook notes cite at this step [4, 6].

Around 250 CE Diophantus studied rational solutions of equations [8, 9]. One of his problems splits 16 into two rational squares, for example $(16/5)^2 + (12/5)^2 = 256/25 + 144/25 = 16$. His work did not grow out of Euclid's; the two lines are independent.

In 1640 Fermat stated, without proof, that if a prime p does not divide a, then p divides a^(p−1) − 1 [10, 11]. A quick check finds no counterexample for any prime $p < 2000$ and any a from 1 to p − 1. Fermat's number theory was inspired by Bachet's 1621 edition of Diophantus [8, 12].

In 1736 Euler gave the first published proof, by expanding (a + 1)^p with the binomial theorem and using induction on a [10, 13, 14, 15]. It came 96 years after Fermat's claim. In 1758 Euler generalised the theorem to $a^{\varphi(n)} \equiv 1 \pmod{n}$ for every a coprime to n, where φ(n) counts the numbers below n that are coprime to it [16, 17]. His proof works with the least power of a that is ≡ 1; the now-familiar proof by permuting residues is Ivory's, from 1806 [14]. A quick check confirms the theorem for every n from 2 to 499 and every a coprime to n. Euler himself called this "a third proof" (E271): it is a new argument, not one that uses the 1736 proof, so there is no arrow of proof use between them [16].

Euler's product and the analytic line

In 1737 Euler proved that $\Sigma\, n^{-s} = \Pi\, (1 - p^{-s})^{-1}$, the sum over all whole numbers equalling the product over all primes, and that $\Sigma\, 1/p$ diverges [18, 20]. A quick check at s = 2: the product over all primes below 10^6, in 30-digit arithmetic, gives

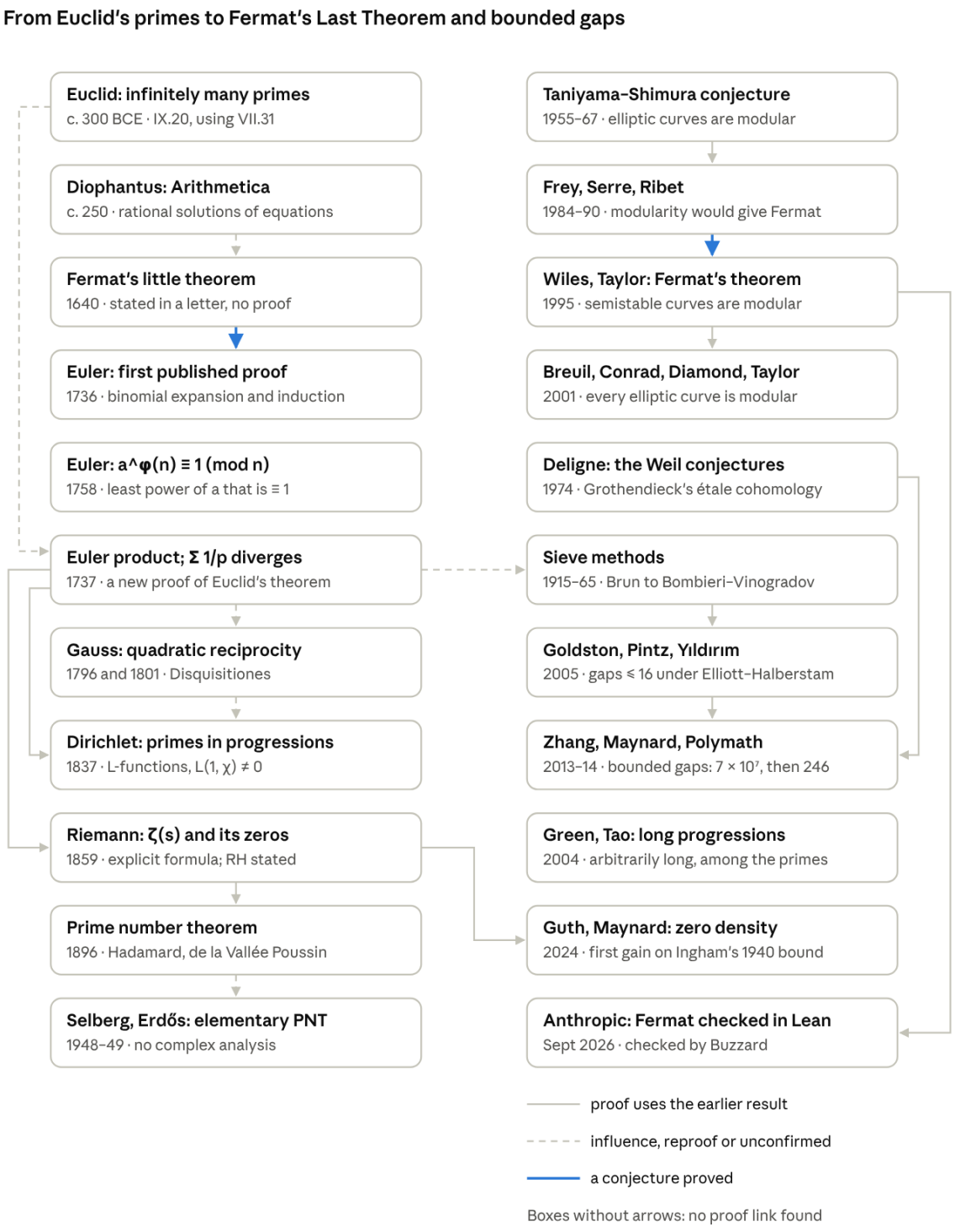


Figure 2: number theory · 22 links from Euclid to Guth–Maynard

1.6449340, against π²/6 = 1.6449341. The formula rests on unique factorisation into primes, which Euler used before Gauss proved it in 1801 [19].

Gauss gave the first proof of quadratic reciprocity in 1796, a long argument by induction on the primes [20, 22, 23]. In the Disquisitiones Arithmeticae of 1801 he made congruences into a concept [19]. The book systematises the work of Fermat, Euler, Lagrange and Legendre, and repairs Legendre's flawed proof of reciprocity [21]. A quick check confirms the law for all 1830 pairs of distinct odd primes below 300.

In 1837 Dirichlet proved that there are infinitely many primes in every progression a + nq with gcd(a, q) = 1 [24, 25, 26]. He followed Euler's product argument, but with L-series in place of ζ. The key step is that L(1, χ) ≠ 0 for every non-trivial character χ [27]. A quick check: for the character mod 4, L(1, χ_{-4}) = 1 − 1/3 + 1/5 − … = π/4. Gauss was an influence on Dirichlet, not a source of proof steps. Dirichlet's theorem also supplies a step that Legendre's reciprocity proof had assumed (note 4).

In 1859 Riemann continued ζ to the whole complex plane and tied its zeros to the count of primes (note 1) [28, 29]. He also stated the Riemann hypothesis, that the non-trivial zeros lie on the line Re s = ½. His own words name the source: "My point of departure is … the observation of Euler." A quick check: the first three zeros lie on the line, since mpmath counts N(26) = 3 zeros up to height 26, and Hardy's Z(t) changes sign three times on (0, 26), at t ≈ 14.135, 21.022 and 25.011.

In 1896 Hadamard and de la Vallée Poussin independently proved the prime number theorem, π(x) ~ x / log x [30, 31, 32]. The key step is that ζ(1 + it) ≠ 0, and the proof rests on Riemann's explicit formula. A quick check: the exact count is π(10^7) = 664,579, while li(10^7) = 664,918, with li the principal-value integral of dt / log t from 0 to x. Legendre had printed an estimate for the count of primes in 1808 (note 4).

In 1948–49 Selberg and Erdős found an elementary proof of the prime number theorem [32, 33]. It reproves the result without complex analysis.

The analytic line reaches the present with Guth and Maynard, who in 2024 proved the zero-density bound N(σ, T) ≤ T^(30(1 − σ)/13 + o(1)) (published in the Annals in 2026) [34, 35]. It is the first gain on Ingham's bound of 1940. As a consequence, almost all intervals of length x^(17/30 + o(1)) contain primes. The result rests on Riemann's link between zeros and primes.

Sieves, gaps and long progressions

Sieve methods grew from the sieve of Eratosthenes between 1915 and 1965 [36, 37, 38, 39]. In 1919 Brun proved that the sum of the reciprocals of the twin primes converges. In 1923 Hardy and Littlewood conjectured the density of prime k-tuples (note 5). Selberg's sieve followed, though the sources give it no date, and then the Bombieri–Vinogradov theorem, which gives primes a level of distribution ½. A quick check: there are exactly 58,980 twin pairs (p, p + 2) with p + 2 < 10^7. Hardy and Littlewood's formula $2C_2 \int_2^{(10^7)} dt / (\log t)^2$, with C_2 = 0.6601618, predicts 58,754.

In 2005 Goldston, Pintz and Yıldırım proved that gaps between primes can be very small compared with the average gap [39, 40]. They also showed that gaps would be bounded if the level of distribution exceeded ½, and at most 16 under the Elliott–Halberstam conjecture. Their method rests on Selberg's weights and the Bombieri–Vinogradov theorem.

The gap problem borrowed a tool from algebraic geometry. In 1974 Deligne proved the Riemann hypothesis for varieties over finite fields [41, 42, 43]. Ramanujan's conjecture |τ(p)| ≤ 2p^(11/2) follows from it. A quick check of the simplest case, Hasse's bound: for the curve $y^2 = x^3 + x + 1$, the value $a_p = p + 1 - \#E(\mathbb{F}_p)$ satisfies $|a_p| \le 2\sqrt{p}$ for every prime $p < 3000$. Deligne's proof rests on Grothendieck's étale cohomology and trace formula. Riemann's hypothesis serves as an analogue only, not as a step.

In 2013 Yitang Zhang proved that infinitely many prime gaps are below 7×10^7 [44, 47]. He used the GPY method with a stronger form of Bombieri–Vinogradov and bounds of Deligne type. Later in 2013 Maynard independently refined the GPY method and got 600 [45]. In 2014 the Polymath project brought the bound to 246 [46]. A quick check: $7 \times 10^7 / 246 \approx 284{,}553$, so the bound fell by a factor of nearly 300,000 in a year. Since August–September 2026, preprints claim gaps of 240 (Stadlmann) [48], 212 (Axiom Math) and 186 (OpenAI) [49]. None is peer-reviewed as of 28 October 2026 (note 6) [50].

In 2004 Green and Tao proved that the primes contain arbitrarily long arithmetic progressions [51, 52]. A quick check: 199 + 210k is prime for every k from 0 to 9, a progression of ten primes. The proof rests on Szemerédi's theorem, a transference principle and the Goldston–Yıldırım majorant (note 6).

Modularity and Fermat's Last Theorem

Between 1955 and 1967 Taniyama, Shimura and Weil shaped the conjecture that every elliptic curve over ℚ is modular [53, 54, 56]. Weil's 1967 paper proposed the conductor as the level, which made the conjecture precise and checkable [56]. A quick check on the curve 11a1, $y^2 + y = x^3 - x^2 - 10x - 20$: the values $a_p = p + 1 - \#E(\mathbb{F}_p)$, found by counting points, match the coefficients of $q \cdot \Pi (1 - q^n)^2 (1 - q^{11n})^2$ for every prime $p < 400$ [55]. The conjecture had no proof behind it at the time.

Between 1984 and 1990 Frey, Serre and Ribet showed that a counterexample to Fermat's Last Theorem would give an elliptic curve that cannot be modular (note 2) [57, 58, 59, 60, 61]. If $A^p + B^p = C^p$, Frey's curve is $y^2 = x(x - A)(x + B)$, with A and B standing for the two p-th powers. A quick check with sympy: this curve has discriminant $16(ABC)^2$, with C = A + B, which is a perfect p-th power up to small factors. So proving the modularity conjecture for semistable curves would prove Fermat.

In 1995 Wiles, with Taylor and Wiles for the crucial step, proved that semistable elliptic curves are modular, and so Fermat's Last Theorem holds [62, 63, 64]. The 1993 version had a gap in a bound for a Selmer group; it was repaired in 1994 with Hecke algebras. The proof rests on Ribet's reduction. It came 358 years after Fermat's margin note of about 1637 (note 3).

In 2001 Breuil, Conrad, Diamond and Taylor extended Wiles's result: every elliptic curve over ℚ is modular [54, 65].

In September 2026 Anthropic completed a formal proof of Fermat's Last Theorem in the Lean proof assistant, and Kevin Buzzard checked it: "it checks out" [66, 67, 68]. It follows Wiles 1995 through the account of Darmon, Diamond and Taylor, and is checked by Lean's kernel. According to Buzzard's Xena blog, the repository's own Frey-curve argument covers exponents $p \ge 17$ and relies on an earlier formalisation

for regular primes for the small exponents; a commenter there questions that step [67]. Buzzard's funded project to formalise the modern proof continues [66].

Notes

1. Riemann's date is 1859 [28]. MacTutor's Hadamard page gives 1851, the year of his thesis, which is an error.
2. Frey's date is uncertain. Wikipedia dates his first remarks to 1982 [58], ETSU to an Oberwolfach talk in late 1984 [59], and his paper is from 1986 [61, 64]; this chapter uses 1984. Ribet proved the epsilon conjecture in 1986 [58, 59], and his paper appeared in Inventiones 100, 431–476 (1990) [60]; the date 1987 sometimes given is not confirmed.
3. Fermat's margin note is dated 1637 by Britannica [69], "around 1637" by ETSU [59], but "about 1630" by MacTutor [64].
4. Euler's dates are consistent across the sources: written in 1736, 1737 and 1758, and printed in 1741, 1744 and 1763 [13]. Legendre's constant 1.08366 was printed in 1808 [21, 32]; that he first conjectured the estimate in 1797/98 rests on Wikipedia only. That Dirichlet's 1837 theorem supplies a step Legendre's reciprocity proof assumed is stated by Wright [23] and Wikipedia [22].
5. Hardy and Littlewood's Partitio numerorum III is Acta Mathematica 44 (1923) in the journal record [37]; Soundararajan cites it as 1922 [39].
6. No arrow is drawn from the GPY work of 2005 into Green–Tao 2004: the Goldston–Yıldırım majorant that Green and Tao use is earlier work, so such an arrow would run backwards in time. Of the 2026 gap preprints, the 212 of Axiom Math is reported only by Science News [50].

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Geometry

Two lines run through geometry. The failed attempts to prove Euclid's parallel postulate turned into hyperbolic geometry. Gauss's curvature, made intrinsic, led through Riemann to the Ricci flow with which Perelman proved geometrisation in 2002–03.

Before and in Euclid

Between about 1800 and 1650 BCE Babylonian scribes used the rule $a^2 + b^2 = c^2$ in tables [1, 2]. The tablet Plimpton 322 lists such triples; its first row has width 1,59 = 119 and diagonal 2,49 = 169 in base 60 [3, 4]. A quick check: $119^2 + 120^2 = 169^2$ exactly. Another tablet, YBC 7289, gives $\sqrt{2} \approx 1;24,51,10$, with an error of -6×10^{-7}. No proof of the rule survives [5].

Around 600 BCE, tradition credits Thales with two statements: the angle in a semicircle is a right angle, and the base angles of an isosceles triangle are equal [6, 7]. This is tradition only; what proofs he himself gave is uncertain.

Around 300 BCE Euclid set out his axioms and postulates and proved geometry from them [8, 9, 10, 11]. Proposition I.47, the theorem of Pythagoras, is proved with congruent triangles. Proposition III.31, "often called Thales' theorem", follows from I.5 and I.32 [9]. Proposition XII.2, that circles are to one another as the squares on their diameters, is proved by exhaustion. Euclid proves Thales' statements anew and uses no proof of his, so there is no arrow of proof use. The Babylonian rule was known much earlier, but Euclid's proof does not use it either.

Measuring curves: Archimedes and Apollonius

Around 240 BCE Archimedes proved that $3\ 10/71 < \pi < 3\ 1/7$, using regular polygons of 96 sides inside and outside the circle (note 4) [12, 13, 14]. A quick check: $96 \sin(\pi/96) = 3.14103$ and $96 \tan(\pi/96) = 3.14271$, and both lie between 3.14085 and 3.14286. The method rests on Eudoxus' method of exhaustion, as used in Euclid's Book XII.

Around 200 BCE Apollonius obtained the ellipse, the parabola and the hyperbola as sections of one and the same cone [15, 16, 17]. His treatment uses Euclid's application of areas and his theory of proportion. That dependency is read from the content of his work; the ETSU notes tie him to Euclid's school rather than naming these tools [15].

The parallel postulate and hyperbolic geometry

Ptolemy in the 2nd century CE and Proclus (c. 411–485) tried to prove Euclid's fifth postulate, on parallels, from the others [18, 19, 20, 21, 22]. All such attempts failed, because each assumed something equivalent to the postulate. They were encouraged by a feature of the Elements: Propositions I.1–28 do not use Postulate 5, which made removing it seem possible [18].

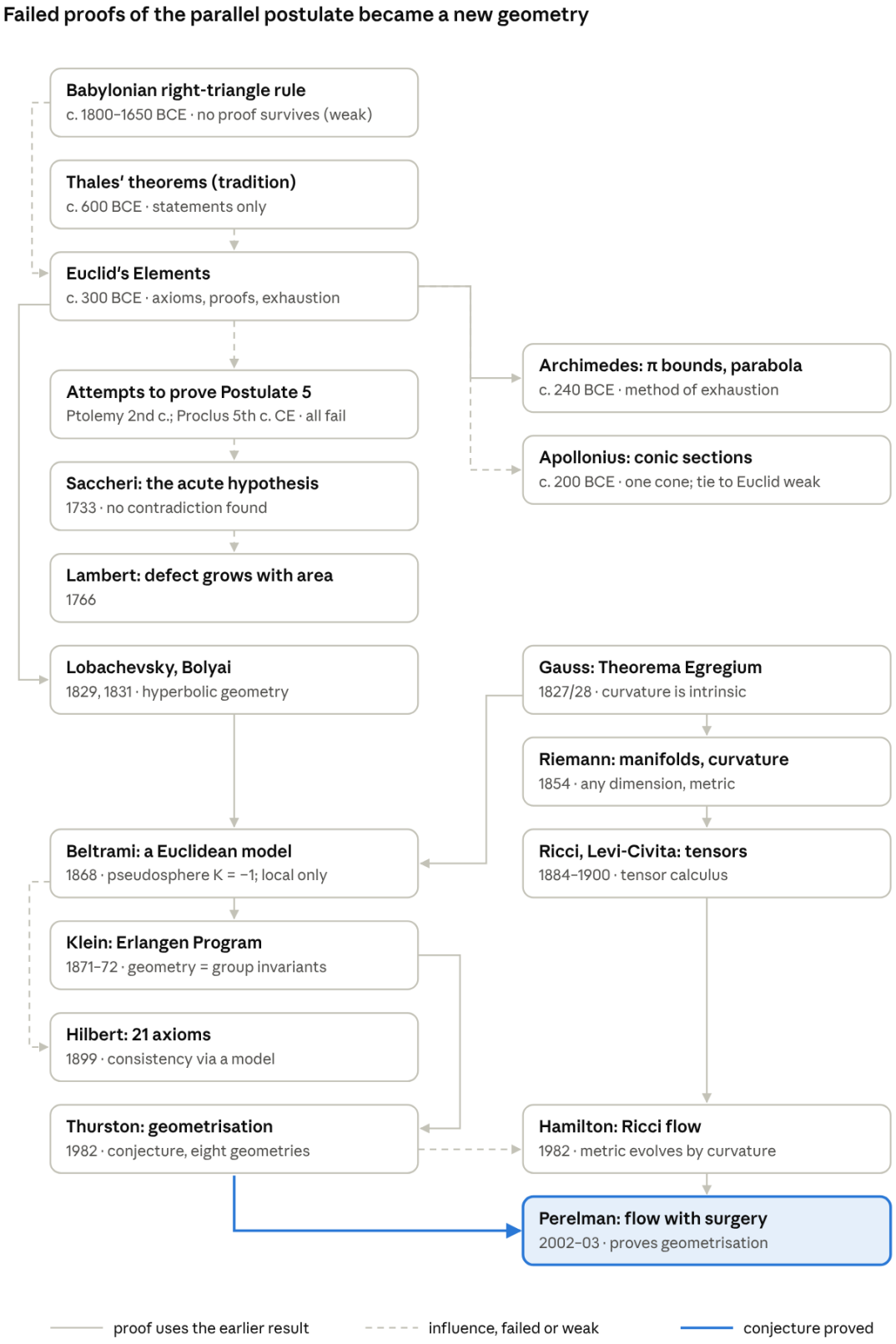


Figure 3: geometry · 18 links from Babylon to Perelman

In 1733 Saccheri denied Postulate 5 and looked for a contradiction (note 3) [23, 24, 20]. Under the hypothesis of the acute angle he found none that was real, and so, without knowing it, proved theorems of hyperbolic geometry. He worked from Euclid's propositions before Postulate 5; according to the Dictionary of Scientific Biography, he took Euclid's first 26 propositions as given [25].

In 1766 Lambert showed that under the acute hypothesis the angle sum of a triangle falls short of 180°, and that the shortfall grows with the area (note 3) [26, 27, 20, 29]. A quick check at curvature -1: the hyperbolic law of cosines gives an equilateral triangle of side 1 angles of 52.64° each, a sum of 157.9°. Lambert came to the problem through Saccheri: Klügel's dissertation of 1763 discussed Saccheri's book and drew Lambert to the parallels problem [28].

In 1829 Lobachevsky and in 1831 János Bolyai published a consistent geometry in which many parallels pass through a point (note 2) [30, 31]. Most of Bolyai's Appendix is "absolute geometry", valid with or without Postulate 5 [30]. Both built on Euclid without Postulate 5. Saccheri's book had little direct influence on them, according to the Dictionary of Scientific Biography, so a failed proof became a theory with no line of dependency between them [25]. According to MacTutor, they never heard of Saccheri [23]. No arrow is therefore drawn from Saccheri or Lambert to Lobachevsky and Bolyai (note 5).

Curvature from Gauss to Klein

On 8 October 1827 Gauss presented his Theorema Egregium, printed in 1828 (note 1) [32, 33, 34]. It says that the curvature of a surface depends only on distances measured inside the surface. A quick check, computed symbolically from the first and second fundamental forms: a sphere of radius R has curvature $1/R^2$, while the plane and the cylinder have curvature 0. The cylinder can be unrolled flat, and its curvature agrees.

In 1854 Riemann generalised Gauss's intrinsic curvature to spaces of any dimension, each with its own metric and curvature (note 1) [32, 19, 35].

In 1868 Beltrami realised hyperbolic geometry on the pseudosphere, a surface of curvature -1, though only locally [36, 31]. In 1870 Houël read this as showing that no construction in the plane can prove Postulate 5 [37]. A quick check: the tractrix surface (sech u cos v, sech u sin v, u − tanh u) has curvature $K = -1$ everywhere, computed symbolically. Beltrami's work rests on Lobachevsky's geometry and Gauss's curvature.

In 1871–72 Klein gave complete projective models of the non-Euclidean geometries [38, 31, 32]. In his Erlangen programme he defined a geometry as what a group of transformations leaves unchanged. His models complete Beltrami's, using Cayley's projective metric.

Tensors, axioms and geometrisation

Between 1884 and 1900 Ricci, later joined by Levi-Civita, built tensor calculus with covariant derivatives (note 4) [39, 40, 41, 42]. Britannica roots tensor analysis in Riemann's geometry [41]. Iurato shows how covariant differentiation was built on Christoffel's ideas [42].

In 1899 Hilbert gave 21 axioms for Euclidean geometry [43, 44]. He reduced their consistency to that of the real numbers, by building a model in the real plane, and showed axioms independent by models in which one axiom fails and the others hold. The work rests on Euclid's system. His model method resembles Beltrami's and Klein's models, but Corry describes it as Hilbert's own and names no such source [45].

In 1982 Thurston conjectured that every closed 3-dimensional space splits into pieces, each carrying one of eight geometries [46, 47, 48]. These eight are model geometries in Klein's sense [48].

Also in 1982 Hamilton introduced the Ricci flow, in which the metric changes in proportion to its Ricci curvature [49, 46]. A quick check: under $\partial g/\partial t = -2\,\mathrm{Ric}$, a round 3-sphere of radius r has $r^2(t) = r^2 - 4t$, solved symbolically, so it shrinks to a point at time $r^2/4$. Hamilton's programme took Ricci curvature as its tool and Thurston's conjecture as its target.

In 2002–03 Perelman completed Hamilton's programme [50, 51, 52]. Ricci flow with surgery proves Thurston's geometrisation, and the Poincaré conjecture with it.

Notes

1. Gauss presented his paper on 8 October 1827, and it was printed in 1828, as its own title page shows [33]. Riemann's 1854 lecture appeared in volume 13 of the Göttingen Abhandlungen, dated 1867 by the Trinity College Dublin edition [53] and 1868 by EuDML [54]; SEP gives 1867 [32], MacTutor 1868 and APS News 1866 [35], so the printing year stays open.
2. For Bolyai's Appendix, MacTutor says his father Farkas had sent Gauss a reprint by 20 June 1831 [30]. The Dictionary of Scientific Biography calls the enclosure a manuscript and dates the Appendix by the Tentamen, 1832 [55]; the conflict stands.
3. MacTutor's non-Euclidean page dates Saccheri's work to 1697, the year of his Logica demonstrativa [19]; his book on the parallel postulate is from 1733. Lambert wrote in 1766; SEP gives 1786, the year of printing [26].
4. Archimedes' Measurement of a Circle is dated c. 240 BCE in Eves (ETSU notes) [14]; an earlier draft had c. 250 BCE. Ricci's calculus is dated 1884–94 by MacTutor [39] and 1887–96 by Britannica [41]. Plimpton 322's first row (119 and 169) is given by Joyce's Clark notes [3] and Wikipedia [4], not by MacTutor or Britannica.
5. The absence of an arrow from Saccheri or Lambert into Lobachevsky–Bolyai rests on the DSB's judgement of little direct influence [25] and MacTutor's statement [23].

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Algebra

Algebra's chain turns on one question: which equations can be solved by a formula? The quadratic rule is used inside the cubic formula, and the cubic inside the quartic. Lagrange's analysis of why these formulas work is what Ruffini, Abel and Galois used to prove that the quintic has none, and Galois's groups became the abstract structures of the 20th century. Khayyam's geometric cubics stand apart: no transmission to Europe is known.

Quadratics from Babylon to Khayyam

Between about 1900 and 1600 BCE Babylonian scribes solved quadratic problems by a procedure equivalent to $x = \sqrt{((b/2)^2 + c)} - b/2$ (note 1) [1, 2, 3, 4]. They gave worked examples, not a general proof. A quick check: for $x^2 + x = 3/4$ the procedure gives $\sqrt{(1/4 + 3/4)} - 1/2 = 1/2$.

Some time after 813 al-Khwārizmī reduced quadratic problems to six normal forms and solved each by a rule [5, 6]. He justified the rules both algebraically and geometrically; the geometric proof completes the square, as MacTutor shows for $x^2 + 10x = 39$ [5]. A quick check: $x^2 + 10x = 39$ gives $(x + 5)^2 = 64$, so $x = 3$. The method is the same as the Babylonian one. Høyrup traces the diagram of this first proof to a practical cut-and-paste tradition of surveyors, on which both the Old Babylonian scribes and the Islamic algebraists drew; he does not claim a direct line from the Babylonian tablets [7].

Around 1070–79 Omar Khayyam found the positive roots of 14 types of cubic equation as the points where two conic sections meet [8, 9, 10]. His methods rest on Euclid (VI.13) and Apollonius (II.12); Berggren confirms that they draw on Euclid's Elements and Data and on Apollonius's Conics [11]. Al-Khwārizmī was an influence: MacTutor writes that turning geometry into equations was "essentially impossible before" him [8].

Formulas for the cubic and the quartic

Between 1515 and 1545 del Ferro, Tartaglia and Cardano solved the cubic [12, 13]. Writing $x = a - b$ turns $x^3 + mx = n$ into a quadratic equation in a^3. A quick check: for $x^3 + 6x = 20$ the formula gives $x = 2$. The method rests on the quadratic rule. There is no arrow from Khayyam, since no transmission of his work to Italy is known.

In 1540 Ferrari solved the quartic, and Cardano printed the solution in 1545 [12, 14, 15]. Once a root of a resolvent cubic is known, the quartic splits into two quadratics. The method rests on the cubic formula.

In 1572 Bombelli gave rules for computing with $\sqrt{-1}$, which extract the real roots hidden in Cardano's formula (note 6) [16, 17]. His work rests on Cardano's formula.

In 1591 Viète used letters for known and unknown quantities alike, and Descartes in 1637 developed the notation further [18, 19]. They studied the relations between the

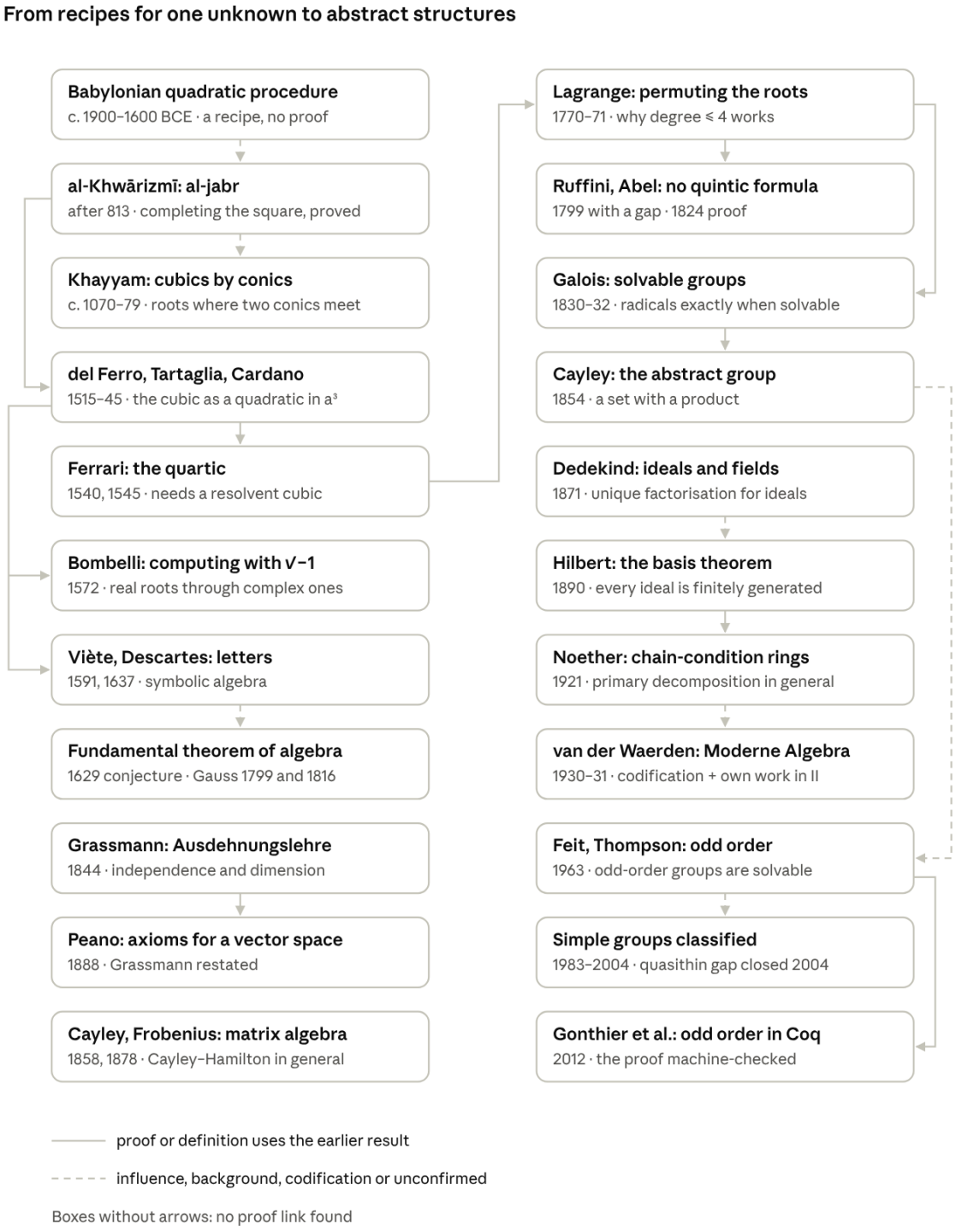


Figure 4: algebra · 21 links from Babylon to Coq

roots of an equation and its coefficients. A quick check: $(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$, where 6 is the sum of the roots, 11 the sum of their products in pairs, and 6 their product. Viète builds on Cardano, and Descartes on Viète [19]. Bombelli's work did not feed into Viète's; the two lines are independent.

The fundamental theorem of algebra

Between 1629 and 1816 the fundamental theorem of algebra was stated and proved [20, 21, 22, 23, 24]. In 1629 Girard conjectured that an equation of degree n has n roots. D'Alembert in 1746 and Gauss in 1799 gave proofs with gaps [23]. Argand's proof of 1806 has, according to MacTutor, "small gaps", while Wikipedia calls it the first rigorous proof (note 4) [24, 21]. Gauss's proof of 1816 was complete; Gamboa and Cowles call his later proofs "all essentially correct" [20, 22]. The theorem needs the symbolic polynomials of Viète and Descartes even to be stated.

Permutations and the quintic

In 1770–71 Lagrange studied how functions of the roots change when the roots are permuted [25, 26]. This explains why the cubic and quartic formulas work. A quick check: $(x_1 + \omega x_2 + \omega^2 x_3)^3$, with ω a cube root of unity, takes only 2 values under the six permutations of three roots, and $x_1x_2 + x_3x_4$ takes only 3 values under the 24 permutations of four roots. The cubic thus reduces to a quadratic, and the quartic to a cubic.

In 1799 Ruffini argued that the general quintic has no formula in radicals, and in 1824 Abel closed the gap in his proof [25, 27, 28, 29]. Both rest on Lagrange's permutation analysis; Rice notes that Ruffini used Lagrange directly [29]. A quick check: the derived series of the symmetric group S_5 stops at the alternating group A_5, with orders 120 and 60, so S_5 is not solvable.

In 1830–32 Galois proved that an equation is solvable by radicals exactly when its group is solvable [30, 27]. A quick check: $x^5 - x - 1$ is irreducible and its Galois group is S_5, so it has no solution in radicals. Galois built on Lagrange. Abel's work overlapped with his, but Galois did not use it; the two are parallel.

Abstract structures

In 1854 Cayley gave the first abstract definition of a finite group, with its multiplication table [31, 32, 33]. He also gave the one-to-one correspondence behind what is now called Cayley's theorem (note 2) [34]. His starting point was the substitution groups of Galois, whom he credits in a footnote: "The idea of a group as applied to permutations or substitutions is due to Galois" [32, 33]. According to Barnett, Cauchy's work on permutation groups moved towards abstraction and so prepared the ground for Cayley's definition; no source shows Cayley using Cauchy's results [35].

In 1871 Dedekind introduced ideals, which restore unique factorisation in rings of algebraic integers (note 3) [36, 37]. They replace Kummer's ideal numbers. A quick check: in $\mathbb{Z}[\sqrt{-5}]$, $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. All four factors are irreducible, since no element has norm $a^2 + 5b^2$ equal to 2 or 3.

In 1890 Hilbert proved that every ideal of the polynomial ring $k[x_1, \ldots, x_n]$ is finitely generated (note 3) [38, 39, 40]. His argument was non-constructive. It answered Gordan's problem on the invariants of binary forms. SEP says that Dedekind shaped Hilbert's algebra in general, but no source opened ties the 1890 theorem itself to Dedekind's ideals [36].

In 1921 Emmy Noether proved that in any ring with the ascending chain condition, every ideal is an intersection of primary ideals [41, 42, 43]. A quick check of the chain condition in the integers: $(12) \subset (6) \subset (3) \subset (1)$ stops after three steps. Her theorem rests on Lasker's result of 1905 for polynomial rings, on Hilbert's chain condition and on Dedekind's ideals [36, 44]. Corry quotes Noether herself: "Es steht alles schon bei Dedekind" (it is all already in Dedekind) [44].

In 1930–31 van der Waerden's Moderne Algebra codified the lectures of Noether and Artin [45, 46]. It is mostly a codification; according to MacTutor, volume II also contains his own work [45].

Grassmann in 1844 defined linear independence and dimension, and Peano in 1888 gave axioms for a vector space [47, 48, 49]. Peano restates Grassmann; the title of his book says so.

In 1858 Cayley set out matrix algebra [50, 51, 52]. He proved the Cayley–Hamilton theorem for 2×2 matrices and checked it for 3×3. In 1878 Frobenius proved it in general, closing Cayley's gap. According to MacTutor, Frobenius did not know Cayley's memoir, so his proof may be independent [50].

Finite simple groups

In 1963 Feit and Thompson proved that every group of odd order is solvable, in 255 pages (note 5) [53, 54, 55]. A quick check: pages 775–1029 of the Pacific Journal of Mathematics make 255 pages. The proof rests on character theory and on Suzuki's work on CA-groups.

The classification of the finite simple groups was deemed complete about 1980, according to Aschbacher, reprinted by MacTutor (note 7) [56, 57, 58, 59]. That was too early. The quasithin case had a gap, which Aschbacher and Smith closed in 2004 [57, 58], 21 years after the announcement of 1983. The classification uses the Feit–Thompson theorem for the case of odd order, the 2-rank 0 case, and its local analysis became the general method.

In 2012–13 Gonthier and others checked the odd order theorem by machine in the Coq proof assistant [60, 55]. Their work follows Feit and Thompson's proof.

Notes

1. The date c. 1900–1600 BCE comes from Stony Brook, which dates the quadratic tablet YBC 6967 to about 1900 BC [3], and Wikipedia, which dates most tablets to 1800–1600 BC [4]; MacTutor and Høyrup say only "Old Babylonian" [1, 2].
2. That Cayley did not show the correspondence to be a homomorphism is stated only by Wikipedia [34].
3. Hilbert's basis theorem is dated 1890, the paper with the proof in Math. Ann. 36, 473–534 [38]. An announcement of 1888 is given only by MacTutor's Hilbert page [40]. The 1893 paper is a second one, on the Nullstellensatz, not the basis

theorem [61]; MacTutor's ring-theory page nevertheless dates the basis theorem 1893 [37]. Dedekind's ideals first appeared in 1871, in the supplements to the second edition of Dirichlet's Vorlesungen [36, 37, 62], but MacTutor's Dedekind page says he introduced them in the 1879 and 1894 editions [63]; this stays a conflict. SEP gives 1893 for the fourth edition, Wikipedia and MacTutor's Dedekind page 1894.

4. Argand's proof is from 1806 [24, 21], republished in Gergonne's Annales in 1813; MacTutor's page on the theorem gives 1814 [20]. "Small gaps" is MacTutor's Argand page only.
5. Feit–Thompson runs to 255 pages, pp. 775–1029 of Pacific J. Math. 13 [53]; MacTutor's 250 is a round figure [54].
6. Both attributions for $x^3 = 15x + 4$ are right. Cardano met it in his letter of 4 August 1539, according to MacTutor only; Bombelli solved it in 1572, giving $x = 4$ [64].
7. Gorenstein's announcement of the classification in 1983 is given only by Wikipedia [56]. The roles of Feit–Thompson as the odd-order case and of its local analysis as the method are also stated only by Wikipedia; Solomon names p-local structure as the method but not as Feit–Thompson's [58]. The 2004 publication of the quasithin case is given by Solomon [58].

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Analysis

Analysis has two stages. Up to Euler it is a chain of methods: exhaustion, indivisibles, the calculus. From 1807 one claim drives it: Fourier's assertion that every function is a trigonometric series. Proving or refuting that claim forced the rigorous integral (Cauchy, Riemann, Lebesgue), uniform convergence, set theory and function spaces. The question was settled for square-integrable functions in 1966. Liu Hui and Madhava reached results of the same kind independently; no line runs from them to Europe.

Exhaustion and its parallels

Eudoxus (408–355 BCE) created the method of exhaustion [1, 2, 3]. To show that a quantity equals X, one proves by a double reductio that it can be neither larger nor smaller. Figures are inscribed until the excess left over is below any given amount. A quick check: for inscribed polygons with 2^n sides in a circle of radius 1, n = 2 to 13, the gap between half-perimeter and π shrinks by a factor tending to 1/4 per doubling (0.256, 0.251, … 0.2500).

Archimedes (287–212 BCE) used exhaustion to prove that a parabolic segment is 4/3 of its inscribed triangle [4, 2]. With 96-sided polygons he showed that 3 10/71 < π < 3 1/7. A quick check: the inscribed and circumscribed 96-gons of radius 1 have half-perimeters 3.141032 and 3.142715. Both results rest on Eudoxus's exhaustion.

In 263 Liu Hui doubled inscribed polygons by a recurrence based on the Gougu rule, the Chinese form of Pythagoras's theorem (note 6) [5, 6]. From the area of the 3072-gon he obtained π ≈ 3.14159. A quick check: the regular 3072-gon of radius 1 has area 3.1415905. His work does not descend from Archimedes; the two are independent.

Around 1400 Madhava found power series for the arctangent, sine and cosine, and computed π to 11 decimals (note 7) [7, 8, 9]. Proofs appear in Jyesthadeva's Yuktibhasa of c. 1530. A quick check: $\sqrt{12}\ \Sigma\ (-1/3)^k / (2k + 1)$, summed over 21 terms from k = 0 to 20, is off from π by 5.8×10^{-12}. According to MacTutor, these series were "rediscovered in Europe" over two hundred years later [7]. No transmission to Europe is shown, so no arrow is drawn.

Towards the calculus

In 1635 Cavalieri published his method of indivisibles [10, 11, 12]. Around 1636–38 Fermat devised adequality, a method for tangents and extrema (note 7) [13, 14]. A quick check of the method: $((a + e)^2 - a^2) / e = 2a + e$, which at e = 0 gives 2a, the slope of x^2 at a. Cavalieri developed Archimedes' exhaustion; this is a lineage of method rather than a step used in a proof [10, 11].

The fundamental theorem of the calculus, that integration is the inverse of differentiation, took shape between 1666 and 1693 [15, 16]. MacTutor calls Newton's tract of October 1666 the first clear statement [15]. Barrow proved a geometric form in his Lectiones geometricae of 1670. Leibniz gave an analytic form



Figure 5: analysis · 19 links from Eudoxus to Lean

in 1686 and a geometric proof in 1693 [16]. Leibniz built on Barrow's geometric version; Fermat was a precursor only.

In 1735 Euler proved that $\Sigma\ 1/n^2 = \pi^2/6$ [17]. In 1748 he made analysis the study of functions and gave the formula e^(ix) = cos x + i sin x [17, 18]. A quick check: $\Sigma\ 1/n^2$ for $n < 10^6$ is 1.6449331, against $\pi^2/6 = 1.6449341$. His work rests on the calculus and on Johann Bernoulli's concept of a function.

Fourier's claim and the rigorous integral

Between 1807 and 1822 Fourier claimed that every function can be written as a trigonometric series [19, 20]. The committee that judged his work in 1811 faulted its rigour. A quick check shows a real difficulty: the partial sums $(4/\pi)\ \Sigma \sin((2k-1)x) / (2k-1)$ of the square wave, with 100, 1,000 and 10,000 terms, all peak at 1.1790 on a grid of 200,001 points. The Gibbs limit is 1.178980, so the overshoot never dies away and convergence is not uniform. Fourier's series had precursors in the trigonometric series of Euler and Daniel Bernoulli.

In 1821–23 Cauchy defined limits and continuity, gave the Cauchy criterion for convergence, and defined the integral of a continuous function as a limit of sums [21, 22, 23, 24]. His "sum theorem", that a series of continuous terms has a continuous sum, is false. Abel gave a counterexample in 1826.

In 1829 Dirichlet proved that the Fourier series of a bounded, piecewise monotone function converges to (f(x−) + f(x+)) / 2 (note 2) [25, 26, 27]. This proves Fourier's claim under conditions. Dirichlet also showed that Cauchy's convergence proof was defective.

In 1854 Riemann defined the integral as a limit of sums, with an oscillation test that decides which bounded functions are integrable (note 3) [28, 29]. This generalises Cauchy's integral. Riemann's paper starts from Dirichlet's.

Uniform convergence and the sets of Cantor

Between 1841 and 1872 Weierstrass introduced uniform convergence, gave the ε–δ definition of a limit, and found a continuous function that is nowhere differentiable [30, 21, 31, 23]. Uniform convergence first appears in his paper "Zur Theorie der Potenzreihen", signed in autumn 1841 but printed only in 1894 [32]; that early date is in doubt (note 4) [33]. He presented the nowhere-differentiable function to the Berlin Academy on 18 July 1872 [31]. A quick check uses $W(x) = \Sigma\ a^n \cos(b^n \pi x)$ with $a = 1/2$ and $b = 13$, so $ab = 6.5$, above Weierstrass's bound $1 + 3\pi/2 \approx 5.71$. Summing 80 terms in 50-digit arithmetic, with $b^n x$ reduced mod 2 exactly in rationals, the difference quotients at $x = 0.3$ grow in size from 19.8 to 7510 as h falls from 10^{-2} to 10^{-5}. Uniform convergence repairs Cauchy's sum theorem; Seidel reached the same idea in 1847 [23].

In 1870–72 Cantor proved that a trigonometric series converging to 0 outside an exceptional set has all its coefficients zero [34, 35]. To allow larger exceptional sets he iterated the derived set, the set of limit points, and this led him to the transfinite ordinals. His work rests on Riemann's paper on trigonometric series, on Weierstrass's notion of a limit point, and on Cauchy sequences.

Measure, function spaces and Carleson’s theorem

In 1902 Lebesgue defined measure and an integral that partitions the range of a function rather than its domain [36, 37]. With his integral, limits and integrals may be exchanged under broad conditions. It generalises Riemann’s integral and, according to MacTutor, builds on Borel and Jordan [36]. Borel’s measure uses countable unions [38], and the fact that countable sets have measure zero, which Borel took from his reading of Cantor [39].

In 1907 Riesz and Fischer proved that the space L^2 is complete [40]. Between 1920 and 1932 Banach set out axioms for complete normed spaces and, with Hahn, proved the Hahn–Banach theorem [41]. A quick check of the L^2 picture: by Parseval, the square wave with coefficients $b_n = 4/(n\pi)$ for odd n has $\Sigma\, b_n^2 = 2$; summed over odd n up to 200,000 it gives 1.999996. This work rests on Lebesgue’s integral and on the study of integral equations by Hilbert, Fredholm and Volterra.

Sobolev in 1936 and Schwartz in 1944–50 created the theory of distributions, defined as continuous linear functionals on test functions (note 5) [42, 43, 44]. Dirac’s δ became rigorous: δ is the functional that sends φ to φ(0). A quick check: for $\varphi = e\hat{}(-x^2)$, $-\int_0\hat{}\infty\, \varphi'\, dx = \varphi(0) = 1$. The theory rests on functional analysis and duality; Sobolev’s work was a precursor.

In 1922–23 Kolmogorov constructed a function in L^1 whose Fourier series diverges almost everywhere (note 5) [45, 46]. This refuted the L^1 form of Lusin’s conjecture of 1913–15.

In 1966 Carleson proved that the Fourier series of every function in L^2 converges almost everywhere [47, 46]. This came 53 years after Lusin’s conjecture of 1913 (note 1). Carleson’s proof rests on Lebesgue’s L^2. Kolmogorov’s example had made most people expect the opposite answer.

In 2025 Becker, van Doorn, Thiele and others formalised Carleson’s theorem in the Lean proof assistant, through a version on metric spaces [48, 49]. The project was completed in July 2025. It follows proofs of Carleson’s type.

Notes

1. Lusin’s conjecture has two dates, each with two sources: 1913 in MacTutor [47] and a Cornell seminar abstract [50], and 1915, in his dissertation, in Wikipedia [46] and the Encyclopedia of Mathematics [51]. The 1913 primary text was not found.
2. Dirichlet’s date is 1829: the paper is in Crelle vol. 4 (1829), pp. 157–169, and is signed January 1829 [25, 27]. MacTutor’s “published in Crelle’s Journal in 1828” does not match the volume [26].
3. Riemann’s paper was written in 1854 and published by Dedekind in vol. 13 of the Göttingen Abhandlungen, which EMIS dates 1867 [28] and the Encyclopedia of Mathematics 1868 [29].
4. NIST OPSF dates uniform convergence to Weierstrass’s autumn 1841 paper on power series [32], but that paper was printed only in 1894. An article in Archive for History of Exact Sciences (2020) calls the 1841 dating “highly questionable”, because the Werke edition revised its terminology; it finds “gleichmäßig” first in lectures of 1870/71 and “im gleichen Grade” from 1861 [33]. The 1841 start therefore remains uncertain.

5. Kolmogorov’s counterexample was built by June 1922 [45] and printed in 1923 [46]. Schwartz’s first publication is from 1945, in Ann. Univ. Grenoble 21 [44].
6. The 3072-gon gives Liu Hui 3.14159 [5, 6]; the value 3.14 is a separate one, from the 96-gon [5]. Leibniz’s analytic form of the fundamental theorem appeared in 1686 [16].
7. The Yuktibhasa is dated c. 1530 by Sarma [9] and c. 1550 by MacTutor [7]. Fermat’s adequality is dated c. 1636 by Wikipedia [13] and to a January 1638 letter to Mersenne by the Garden of Archimedes [14].

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Topology

Topology grew from Euler's polyhedron formula, whose flawed proof and counterexamples revealed a quantity that does not change under deformation. Poincaré turned it into homology and posed his conjecture in 1904. It was proved dimension by dimension, ending with Perelman in 2002–03.

Euler's formula and its repairs

In 1736 Euler proved that no walk crosses each of the seven bridges of Königsberg exactly once (note 1) [1, 2]. A walk uses two bridges each time it passes through a land mass, so every land mass except the start and the end must touch an even number of bridges. A quick check: the four land masses touch 5, 3, 3 and 3 bridges, all odd, so no such walk exists.

In 1750–51 Euler found that $V - E + F = 2$ for convex polyhedra, with V vertices, E edges and F faces; the papers were printed in 1758 (note 1) [3, 4, 5, 6]. His proof, by cutting off pieces of the solid, was flawed. A quick check: the formula gives 2 for all five Platonic solids, from the tetrahedron ($4 - 6 + 4$) to the icosahedron ($12 - 30 + 20$). The formula shares the idea of the Königsberg paper, that only connections matter, but it does not use that paper as a step in its proof.

In 1794 Legendre gave the first complete proof [3, 4]. He projected the polyhedron onto a sphere, where the angle excess of a spherical triangle equals its area. A quick check: an octant of the unit sphere has three right angles, so its excess is $3\pi/2 - \pi = \pi/2$, which is $4\pi/8$, an eighth of the sphere's area. This repaired the gap in Euler's proof. In 1811 Cauchy gave other proofs, all incomplete.

In 1813 Lhuilier showed that the formula fails for solids with holes, and that the correct value is $2 - 2g$ for a solid with g holes [1, 7]. A quick check: a picture frame made of four bars of square section has $V = 16$, $E = 32$ and $F = 16$, so $16 - 32 + 16 = 0 = 2 - 2 \cdot 1$. This counterexample to Euler's formula pointed to the quantity that does not change under deformation.

Surfaces and connectivity

In 1851 and 1857 Riemann classified surfaces by the number of cuts needed to make them simply connected [1, 8]. He came to this from complex analysis, independently; no source links his work to Lhuilier's.

Between 1847 and 1865 Listing and Möbius studied one-sided surfaces, and Listing coined the word "topology" [9, 10]. Listing's Census of 1861 generalises Euler's formula (note 2) [9, 11]. Gauss's teaching was an influence; Flament writes that Gauss "pushed Listing" into these studies [11].

In 1871 Betti defined connectivity numbers in any dimension [12, 1, 13]. He extended Listing's connectivity to n dimensions after talks with Riemann on connectivity in 1863 [1, 12]. Riemann's invariants of 1857 came first [13]. Betti's numbers are a generalised definition, not a step in a proof.

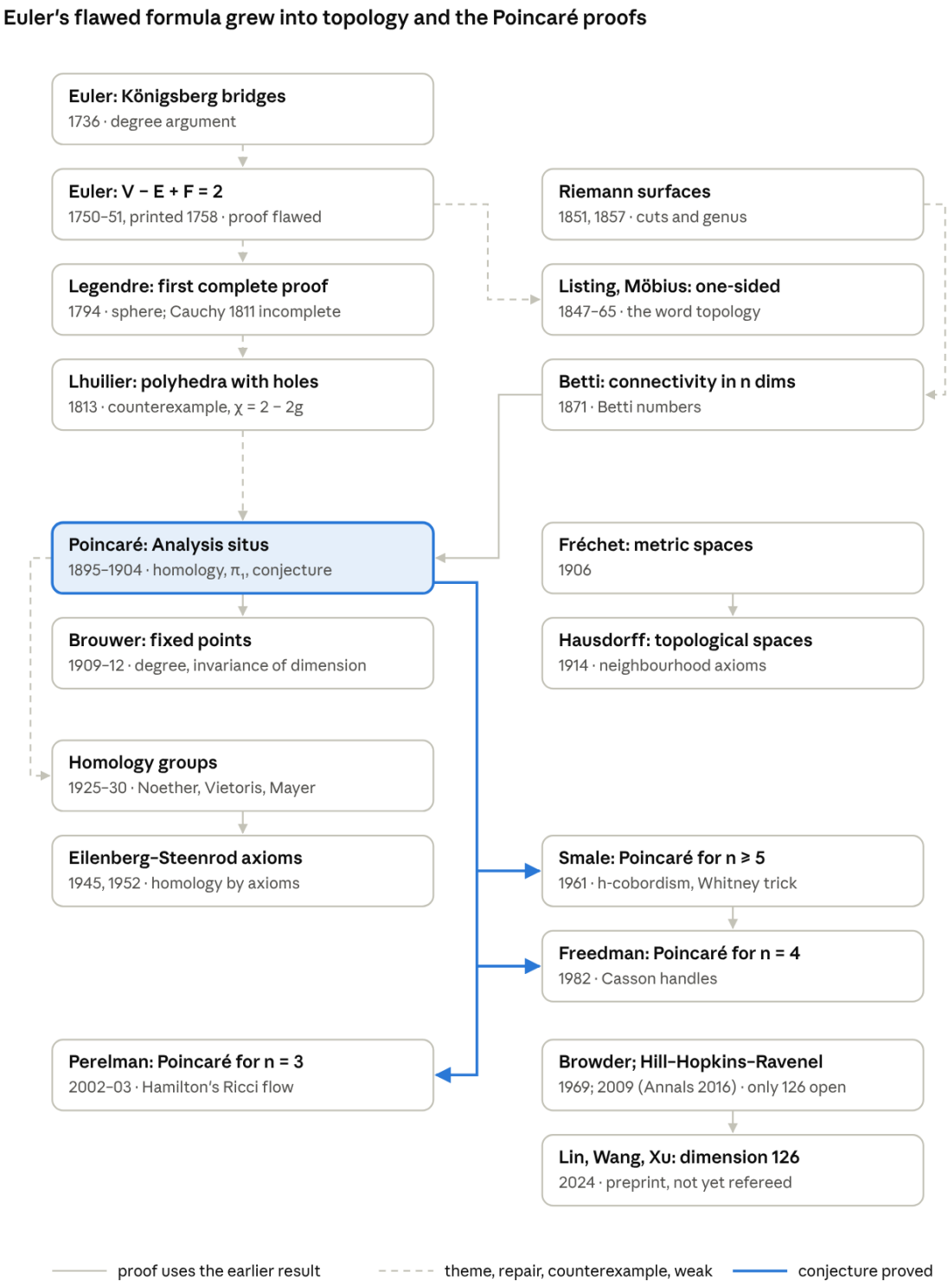


Figure 6: topology · 16 links from Euler to dimension 126 · the diagram draws Fréchet and Hausdorff as two boxes and adds a box for Browder and Hill–Hopkins–Ravenel, on which the Lin–Wang–Xu row rests

In 1895–1904 Poincaré created homology, made Betti's numbers rigorous, introduced the fundamental group, and placed Euler's formula in a general setting [1, 14]. In this way Euler's χ was generalised to any dimension. A counterexample to one of his own claims led him in 1904 to his conjecture: a closed 3-dimensional space in which every loop can be shrunk to a point is a sphere.

Spaces, fixed points and homology groups

In 1909–12 Brouwer proved that every continuous map of a ball to itself has a fixed point, and that dimension is a topological invariant [1, 15, 16, 17]. A quick check in dimension 1: the map cos sends the interval from 0 to 1 into itself, and $\cos x = x$ at $x = 0.7390851$. According to MacTutor, Brouwer made Poincaré's methods into a complete theory [1]. His dimension theory also builds on Poincaré [17].

Fréchet defined metric spaces in 1906, and Hausdorff topological spaces, by neighbourhood axioms, in 1914 [18, 19]. Hausdorff built on Fréchet.

Between 1925 and 1930 Noether, Vietoris and Mayer turned Betti numbers into groups [13, 20, 21]. This reformulated Poincaré's homology in algebraic terms.

In 1945 Eilenberg and Steenrod characterised homology by axioms, and in 1952 they published the full account [22, 23]. The axioms abstract what the existing homology theories had in common.

The Poincaré conjecture, dimension by dimension

In 1961 Smale proved the Poincaré conjecture in dimension 5 and higher, by cancelling handles (note 3) [24, 25, 26, 27]. The proof rests on Morse theory and on the Whitney trick, which needs dimension at least 5 [24, 25, 26]. Milnor's lectures present it as cancelling pairs of critical points [27].

In 1982 Freedman proved the topological Poincaré conjecture in dimension 4 (note 3) [28, 29]. He followed Smale's method, with Casson handles in place of the Whitney trick, which fails in dimension 4.

In 2002–03 Perelman proved the conjecture in dimension 3 [30, 31, 32, 33]. His proof rests on Hamilton's Ricci flow, described in the chapter on geometry.

Kervaire invariant one

In 2024 Lin, Wang and Xu proved that a manifold of Kervaire invariant one exists in dimension 126, the last open case [34, 35, 36]. The dimensions in which such manifolds exist are therefore exactly 2, 6, 14, 30, 62 and 126. The result rests on Browder's theorem of 1969, which tied the problem to dimensions $2^k - 2$, and on Hill, Hopkins and Ravenel, who ruled out all dimensions above 126 (note 3) [37, 38]. On 29 October 2026 it was still an arXiv preprint with no journal version: version 2 dates from 22 February 2025, Xu's publication list gives no journal, and a 2026 paper cites it as a preprint [34, 39, 40].

Notes

1. Euler’s Königsberg paper was presented on 26 August 1735, is dated 1736 and was printed in 1741 [2]. Euler stated his polyhedron formula in a 1750 letter to Goldbach [4]. The two papers were written in 1750 and 1751 [5, 6] and appeared in Novi Commentarii 4, the volume for 1752–53, printed in 1758 [4, 5]. Eppstein’s “published 1752” is the volume’s year [3].
2. Listing’s Census is dated 1861 by Flament [11] and 1862 by MacTutor [9].
3. Smale announced his proof in 1960 in Bull. AMS 66 [41] and published it in Annals 74 (1961) [42]; 1962 is his later paper on h-cobordism. Freedman proved his result at the end of summer 1981 [28] and published it in J. Differential Geom. 17 (1982) [42]. Hill–Hopkins–Ravenel was posted in August 2009 and printed in 2016 [37, 38].

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Probability and statistics

Probability is a chain of ever sharper proofs of one fact: averages settle down. Bernoulli proved it for coin tosses in 1713, and de Moivre, Laplace, Chebyshev and Lyapunov sharpened it. Kolmogorov put it on measure theory in 1933, and everything since is built inside that frame.

Games of chance and expectation

Around 1563 Cardano defined probability as the number of favourable cases over the number of equally possible cases [1, 2, 3]. A quick check of the kind of problem he studied: the chance of at least one six in four throws of a die is $1 - (5/6)^4 = 671/1296 = 0.518$.

In 1654 Pascal and Fermat solved the problem of dividing the stakes of an interrupted game fairly [4, 5, 6]. Pascal's recursion, which values each position by the average of the positions that can follow, is the first expected value. A quick check: if one player needs 1 more win and the other needs 2, the first loses only if the second wins the next two rounds, so his fair share is $1 - 1/4 = 3/4$. They worked on the same dice problems as Cardano, but his book was printed only in 1663 [1], so they could not have used it.

In 1657 Huygens made expectation a rule: p chances of gaining a and q chances of gaining b are worth $(pa + qb)/(p + q)$ [7, 8]. He had heard of the Pascal–Fermat problems from colleagues and re-derived the solutions himself [8]. His work is a reproof, not a step taken from theirs.

The law of large numbers and the normal curve

In 1713 Jacob Bernoulli proved the law of large numbers: the observed frequency of an event approaches its probability as the number of trials grows [9, 10]. A quick check, with $p = 3/5$ and a margin of $1/50$: the exact binomial chance that the frequency lies in the closed interval $p \pm 1/50$ reaches 1000 to 1 for every $n \geq 6520$, scanning n from 1,000 to 26,000. It is not monotone just below that point. Bernoulli's own bound was $n = 25{,}550$, which the Encyclopedia of Mathematics states for the mirror case $p = 2/5$ (note 3) [10]. Part I of Bernoulli's book reprints and annotates Huygens [9], but the proof of the law does not use it.

In 1733 de Moivre showed that the binomial distribution approaches the normal curve (note 2) [11, 12]. A quick check: for 3,600 fair tosses, the exact chance of 1,770 to 1,830 heads is 0.691. The normal value for ± 1 standard deviation, without continuity correction, is 0.683; de Moivre's own figure is 0.682688 [12]. According to MacTutor, he improved on Bernoulli's law, but without using its proof [11]. He used an approximation of $n!$ that involves Stirling's constant $\sqrt{(2\pi)}$ [11, 12].

In 1763 Bayes showed how to go from observed successes back to a distribution for the unknown chance p (note 2) [13, 14, 15]. This inverts the binomial model. Condorcet cited Bayes in his introduction to the volume that contained Laplace's 1781 memoir

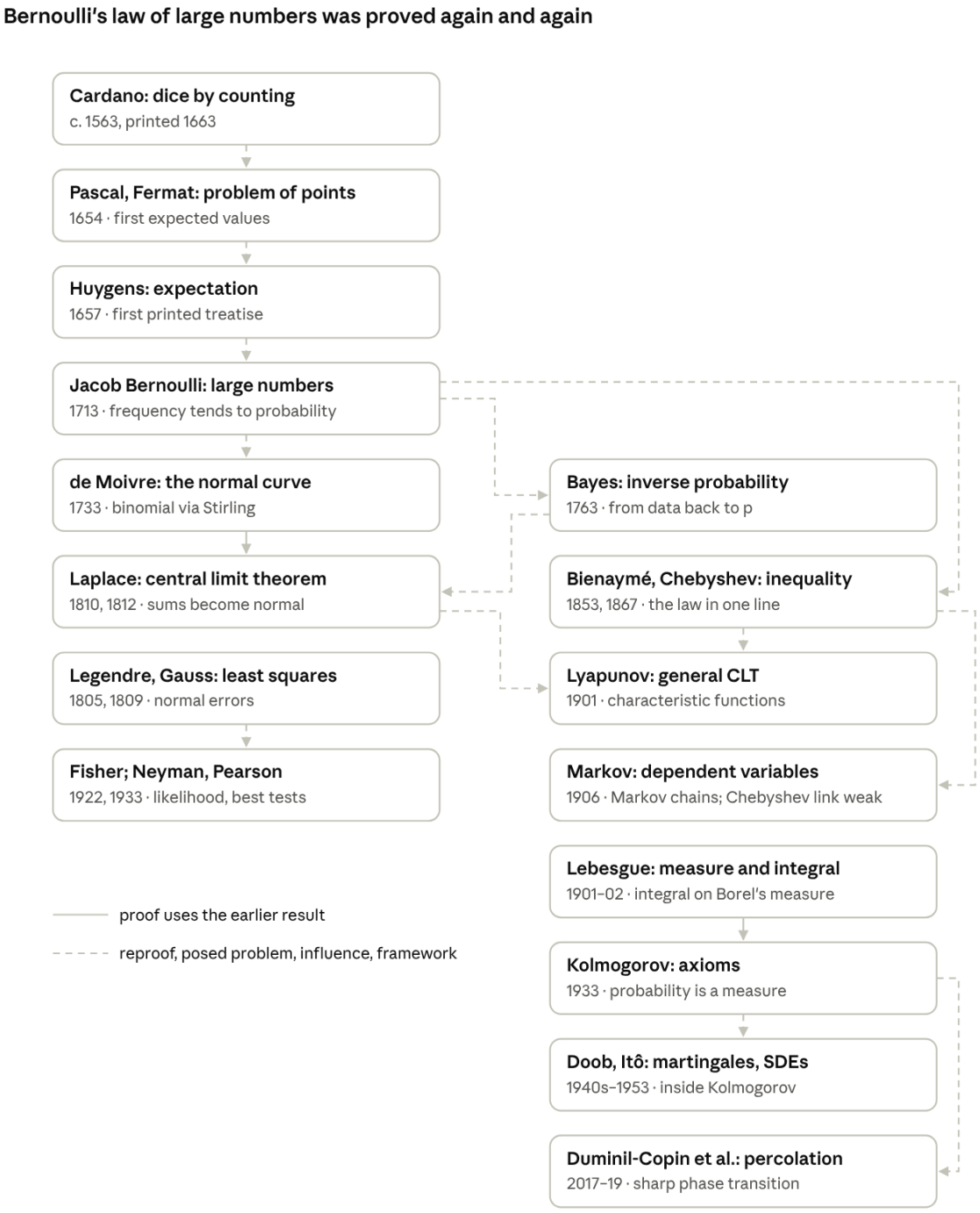


Figure 7: probability and statistics · 16 links from Cardano to percolation

[14, 15]. According to MacTutor, “Bayes’s conclusions were accepted by Laplace” in that memoir; no second source was found for this [13].

In 1810 Laplace proved the central limit theorem, that a sum of many small errors is normally distributed, and set it out in 1812 [16, 17, 18]. It generalises de Moivre’s result; Syed describes Laplace expanding on de Moivre’s binomial-to-normal work [17].

Least squares

Legendre in 1805 and Gauss in 1809 introduced the method of least squares (note 2) [19, 20, 18, 21]. Gauss derived it from the assumption that errors are normally distributed. A quick check: for a random 50 × 3 system (NumPy default_rng, seed 0, standard normal entries), the normal equations $A^{T}Ax = A^{T}y$ agree with numpy.linalg.lstsq within default tolerances. According to Stigler, Gauss adopted Laplace’s analytical set-up of 1774 with a uniform prior [19]. Laplace’s limit theorem of 1810 justified the normal errors only afterwards, so it is background, not a step in Gauss’s proof (note 1) [18, 21].

Sharper limit theorems

Bienaymé in 1853 and Chebyshev in 1867 proved that $P(|X - \mu| \geq k\sigma) \leq 1/k^{2}$ [22, 23]. This inequality proves the law of large numbers in one line, giving a new proof of Bernoulli’s law.

In 1901 Lyapunov proved the central limit theorem for independent terms that need not be identically distributed [24, 25]. He used a new method, characteristic functions, and so generalised de Moivre and Laplace [24, 25]. Chebyshev had posed the problem in 1887 [25].

In 1906 Markov extended the law of large numbers to dependent variables, through what are now called Markov chains [26, 27]. In 1913 he applied the idea to the text of Eugene Onegin [26]. A quick check from his counts in the first 20,000 letters (8,638 vowels; 1,104 vowel–vowel pairs and 15,069 mixed pairs): P(vowel after vowel) = 0.128 and P(vowel after consonant) = 0.663. The stationary share of vowels is then 0.432, matching his count of 8,638 out of 20,000. Markov’s work extends Chebyshev’s law to dependent variables [26, 27]; no source opened says that the proof uses Chebyshev’s inequality.

Measure-theoretic probability and after

In 1901–02 Lebesgue built an integral on Borel’s countably additive measure, as described in the chapter on analysis [28, 29].

In 1933 Kolmogorov defined probability as a measure of total mass 1, with expectation as an integral [28, 30]. By his own account, this rests on Lebesgue’s measure and integral, freed from geometry by Fréchet’s abstract integral [28, 30].

In 1922 Fisher introduced maximum likelihood, and in 1933 Neyman and Pearson proved that the likelihood-ratio test is the most powerful test [19, 31, 32, 33, 34]. Both grew from the estimation methods of Gauss and Laplace. Fisher inspired Neyman [31].

Neyman and Pearson took Fisher’s fixed levels of 5% and 1% and asked where his test statistics came from, though Neyman rejected Fisher’s idea of likelihood [33].

From the 1940s to 1953 Doob developed martingales and Itô the stochastic integral [30, 35]. A quick check: a simulation with NumPy default_rng seed 0, T = 1 and 100,000 steps gives the Itô sum $\Sigma\, W\, dW$ within 1.3×10^{-4} of $(W_T^2 - T)/2$. Both work inside Kolmogorov’s measure-theoretic frame. Itô’s integral realises the ideas of Kolmogorov and Feller path by path [35], and Doob’s martingales rest on Lebesgue integration [30]. The dependence is on a framework, not on a single result.

In 2017–19 Duminil-Copin, Raoufi and Tassion proved sharp phase transitions for a wide class of percolation models [36, 37]. Their method generalises a decision-tree inequality of O’Donnell, Saks, Schramm and Servedio (OSSS). Duminil-Copin received the Fields Medal in 2022 [37].

Notes

1. An earlier version of this chain said that Laplace’s limit theorem “then justified” Gauss’s normal errors, as if it were a step in his proof. Laplace’s theorem is from 1810, after Gauss’s derivation of 1809, so it is background only, and no arrow runs from it to least squares. Laplace’s set-up of 1774, which Gauss adopted, is not drawn as a separate link.
2. De Moivre’s Latin pamphlet is of November 1733, and the English version is in the second edition of 1738 [12]. Bayes’s essay was read on 23 December 1763 and printed in 1764, in the volume for 1763 [14]. Legendre’s appendix on least squares is in the first edition of 1805; 1806 is the second edition [20].
3. Bernoulli’s figure of 25,550 is given in the Encyclopedia of Mathematics and follows from Bernoulli’s own rule [10].

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Combinatorics and graph theory

Combinatorics is the least connected of the chains: several of its classic results stand alone. Two lines are clear. The colouring line runs from Euler's formula through Kempe's flawed proof to the computer proof of 1976. The Ramsey line runs from the bounds of 1935 and 1947 to their first improvements in 2023 and 2025.

Counting before graphs

Around 200 BCE Piṅgala showed that poetic metres of n syllables, each long or short, number 2^n, and counted them by recursion [1, 2]. A quick check: for n = 6, the numbers of metres with 0, 1, … 6 long syllables are 1, 6, 15, 20, 15, 6, 1, which add up to $64 = 2^6$.

From about 1000 to 1654 the arithmetic triangle of binomial coefficients appeared in the work of al-Karaji, Jia Xian and Yang Hui [3, 4, 5]. Pascal proved its properties by explicit induction in 1654 [4]. Piṅgala's sutra is very terse. According to Kulkarni, it is only through Halāyudha's commentary that it can be read as the meru, a figure resembling the triangle (note 1) [6]. Shah disputes that Piṅgala himself had it [1].

In 1736 Euler solved the Königsberg bridge problem by the parity of degrees, and in 1750 he stated $V - E + F = 2$ in a letter to Goldbach [7, 8, 9, 10]. He wrote the formula up in 1750–51, and it was printed in 1758 in the volume for 1752–53. Both results are treated in the chapter on topology.

Between 1740 and 1748 Euler studied partitions by generating functions [11, 12, 13]. He proved that the number of ways to write n as a sum of distinct parts equals the number of ways to write it as a sum of odd parts. A quick check confirms the equality for every $n \le 40$. This result stands alone in the chain.

In 1847 Kirchhoff showed how to count the spanning trees of a network with a determinant [14, 15]. Cayley's formula, that there are n^{n-2} labelled trees on n points, is proved today by Kirchhoff's theorem [15]. A quick check: for n = 8 the determinant gives $262{,}144 = 8^6$. The history of the formula is older than Cayley's note of 1889 (note 3) [14]. Sylvester stated the count without proof in 1857, and Borchardt gave the first proof, by determinants, in 1860. Cayley worked through n = 5 and n = 6 and said that the argument generalises.

The colouring line

In 1879 Kempe published a proof that four colours suffice for any map [16, 17]. In 1890 Heawood found a gap in it, and proved that five colours suffice, reusing Kempe's chains of alternating colours. Both rest on Euler's formula: for a map drawn as a planar graph, $E \le 3V - 6$, so some country has at most 5 neighbours.

In 1976 Appel and Haken proved that four colours suffice (note 2) [16, 18]. They found an unavoidable set of configurations, each of them reducible, and checked reducibility by computer. In 1996 the proof was simplified, with 633 configurations [18]. The work

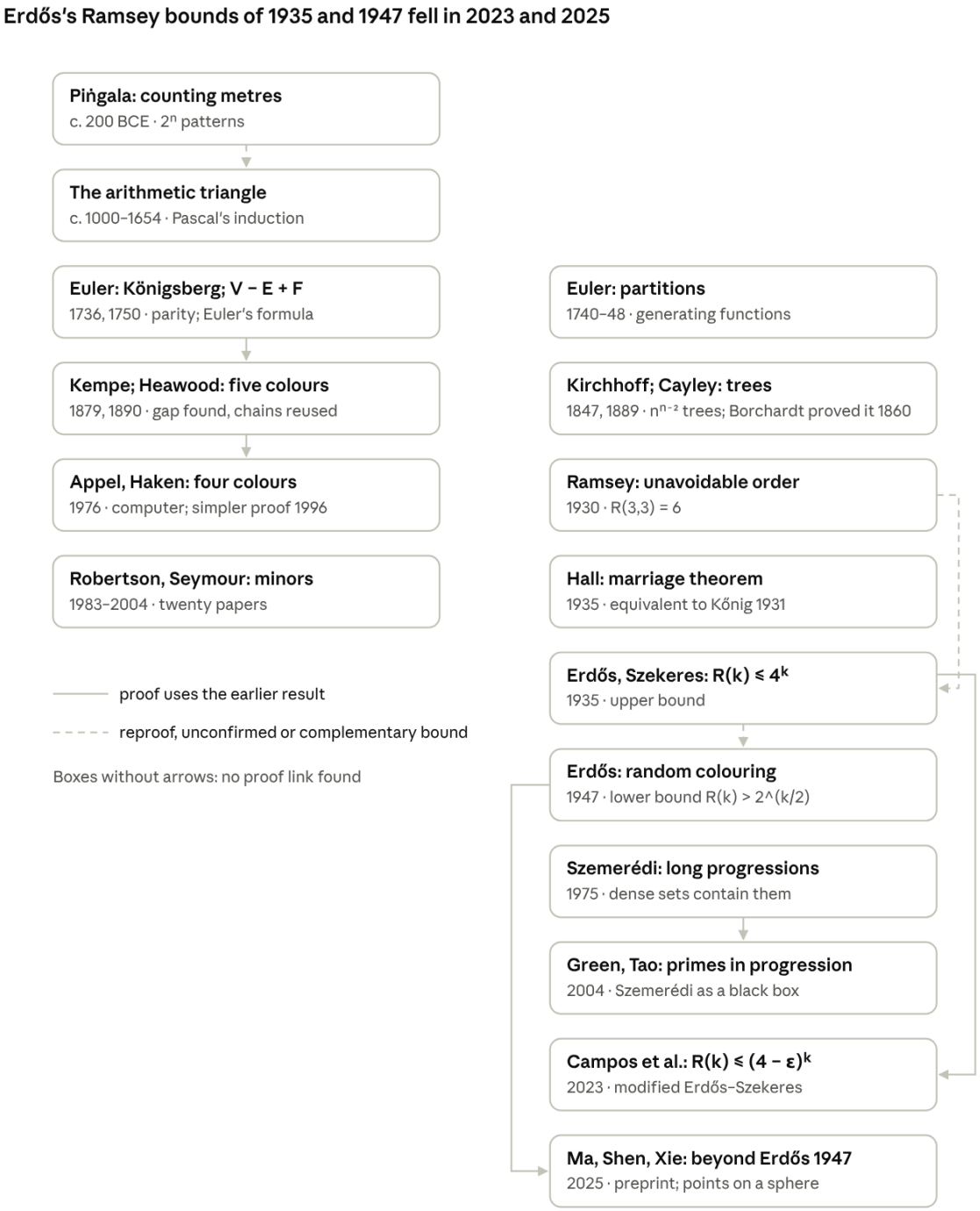


Figure 8: combinatorics and graph theory · 16 links from Piṅgala to 2025

rests on Birkhoff's notion of reducibility and Heesch's method of discharging, both of which come from Kempe's line of argument.

The Ramsey line

In 1930 Ramsey proved that any large enough structure contains an ordered part (note 4) [19, 20, 21]. A quick check of the smallest case: all $2^{15} = 32{,}768$ two-colourings of the 15 edges between 6 points contain a triangle of one colour. With 5 points this can fail: colour the sides of a pentagon red and its diagonals blue.

In 1935 Philip Hall proved that a matching exists exactly when every group of k people has at least k possible partners between them [22, 23]. The theorem is equivalent to the results of Kőnig and Egerváry from 1931, but it was not proved from them [22].

Also in 1935 Erdős and Szekeres proved the upper bound $R(k) \le C(2k - 2, k - 1) < 4^k$ for the diagonal Ramsey numbers [24, 20, 25]. Their argument is a new proof of Ramsey's theorem.

In 1947 Erdős proved the lower bound R(k) > 2^(k/2) by colouring the edges at random [20, 26]. This was the start of the probabilistic method. It answers the other side of the same question and does not use the upper bound.

In 1975 Szemerédi proved that every set of whole numbers of positive density contains arbitrarily long arithmetic progressions [27, 28]. This proved the conjecture of Erdős and Turán from 1936.

Between 1983 and 2004 Robertson and Seymour proved that graphs are well-quasi-ordered by minors [29, 30, 31]: in any infinite list of graphs, one is a minor of a later one. Kuratowski's characterisation of planar graphs and Wagner's conjecture motivated the work [29]. They were a motivation, not a step in the proof.

In 2004 Green and Tao proved that the primes contain arbitrarily long arithmetic progressions [32, 33]. A quick check: 199 + 210k is prime for every k from 0 to 9. The proof uses Szemerédi's theorem as a black box and transfers it to the primes.

In 2023 Campos, Griffiths, Morris and Sahasrabudhe proved that $R(k) \le (4 - \varepsilon)^k$ for some fixed $\varepsilon > 0$, the first exponential improvement on the bound of 1935 [34, 35, 26]. Their method modifies the Erdős–Szekeres procedure: it first builds a red "book" instead of a clique [26]. The paper appeared in the Annals in 2026 [34].

In 2025 Ma, Shen and Xie gave the first exponential improvement on Erdős's lower bound of 1947 for off-diagonal Ramsey numbers [36, 37]. They kept Erdős's random method but used random points on a sphere instead of coin flips. The result is a preprint, not yet peer-reviewed.

Notes

1. Kulkarni places Halāyudha's commentary in the 13th century [6]; other sources place him in the 10th.
2. Appel and Haken's configuration counts belong to different versions of the set. The figure 1,936 is the June 1976 announcement [38, 39, 40], and Rutherford agrees [17]. The figure 1,482 is from the 1977 papers in the Illinois Journal of Mathematics [41, 40]. The figure 1,476 is a later reduced set [18, 39]. No source

opened dates the 1,476 set. MacTutor gives about 1,500 [16], and 1,834 appears only in Wikipedia [42].

3. Sylvester stated Cayley's formula without proof in 1857 and Borchardt proved it in 1860, before Cayley's note of 1889 [14].
4. Ramsey's paper was received on 28 November and read on 13 December 1928, and printed in Proc. London Math. Soc. (2) 30, 264–286, in 1930 [21, 43, 19]. MIT 18.226 gives 1929 [20].

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Differential equations and dynamical systems

This field has more than one root, and several of its famous links turn out not to be proof dependencies. Newton's planetary problem leads through celestial mechanics to Poincaré's discovery of chaos. The line of partial differential equations runs separately, from d'Alembert's vibrating string to the 2025 derivation of the fluid equations from particles. In all, the chain has nineteen links, from Kepler to 2025.

From planetary laws to the equations of motion

Kepler found his planetary laws in 1609 and 1619, from Tycho Brahe's observations [1, 2]. They start the story but rest on no earlier proof.

In 1687 Newton proved that an inverse-square force gives conic orbits and sweeps out equal areas in equal times. A quick check: with GM = 1, orbits integrated numerically (relative and absolute tolerance 10^{-12}) for (a, e) = (1, 0), (1, 0.5) and (2, 0.3) all give $T^2/a^3 = 39.478 = 4\pi^2$. Kepler's rules entered Newton's work as observed phenomena, as background rather than as a proof step [3, 2]. Book 1, Propositions 1–2 prove that equal areas hold if and only if the force is central, and the 3/2 rule follows from the inverse-square law. The Stanford Encyclopedia calls the view that Newton built on Kepler's laws "wrong on several counts", since by 1684 they were known to hold only "to high approximation" [3].

Between 1690 and 1739 the Bernoullis and Euler developed methods for ordinary differential equations. In 1736 Euler wrote Newton's dynamics as differential equations. In a letter to Johann Bernoulli of 15 September 1739 he integrated linear equations with constant coefficients [4, 5, 6, 7]. This restates Newton's mechanics in the language of analysis.

From 1767 to 1799 Euler, Lagrange and Laplace found special solutions of the three-body problem, and Laplace introduced the equation that bears his name. A quick check: $\Delta(1/r) = 0$ for $r = \sqrt{(x^2 + y^2 + z^2)} \neq 0$, so the Newtonian potential satisfies it. This work rests on Newton's law of gravitation [8, 9].

The partial differential equations of physics

In 1747 d'Alembert showed that the wave equation of a vibrating string has the solution $F(x - ct) + G(x + ct)$; substituting this into the equation gives 0 identically [10, 11]. It rests on no earlier link in this chain.

Fourier solved the heat equation by trigonometric series, in his memoir of 1807 and his book of 1822 [12, 13]. A quick check with the square wave sign(sin x): the partial sums $(4/\pi) \Sigma \sin(kx)/k$ over odd k, with 10, 100, 1,000 and 10,000 terms, give 0.968, 0.9968, 0.99968 and 0.999968 at $x = \pi/2$. So they converge to the function except at the jumps, where every partial sum is 0, the midpoint. The trigonometric series of Daniel Bernoulli and Euler were a precedent here, not a proof dependency.

In 1836–37 Sturm and Liouville studied eigenvalues and the zeros of eigenfunctions [14, 15]. A quick check: for $-y'' = \lambda y$ on $[0, \pi]$ with $y(0) = y(\pi) = 0$, finite differences on 2,000 interior points give $\lambda = 1, 4, 9, 16, 25$ to four digits. Their questions came from the theory of heat.

From the 1820s to the 1890s Cauchy, Lipschitz and Picard proved that $y' = f(x, y)$ has a unique solution. Picard's method is now called Picard iteration. According to Bufetov and Zavolokin, Picard first introduced the iterative method for a partial differential equation and then applied it to ordinary ones (note 1) [16, 17, 18, 19]. A quick check: for $y' = y$, the sixth approximation is exactly the Taylor polynomial of e^t of degree 6. Picard replaced Cauchy's assumption by the Lipschitz condition [16]. Existence and uniqueness under that condition is due to Lipschitz in 1868 [19].

Cauchy in 1842 and Kovalevskaya in 1874 proved that analytic problems for partial differential equations have analytic solutions; Kovalevskaya's theorem generalises Cauchy's special case [20, 21]. The form of the equation matters. For the heat equation $u_t = u_{xx}$ with $u(0, x) = 1/(1 - x)$, the power series at $x = 0$ is $\Sigma\ (2n)!/n! \cdot t^n$. Its terms grow once n exceeds about $1/(4t)$; at $t = 0.01$ the term with $n = 500$ is about 10^{433}. So the series diverges for every $t \neq 0$, which is why the theorem requires the Kovalevskaya form.

Stability, chaos and the shape of orbits

In 1890 Poincaré showed that the three-body problem has no further analytic first integrals (note 2) [22, 8, 23]. Holmes states it as no integrals "independent of energy" where transverse homoclinic points exist [23]. Poincaré also found that homoclinic points produce chaotic motion; in Holmes's words they "imply that 'chaotic motions' exist nearby" [22, 23]. This rests on Newton's three-body problem.

In 1892 Lyapunov proved stability by means of a function V that decreases along solutions (note 3) [24, 25]. A quick check: for the damped pendulum $\theta' = \omega$, $\omega' = -\sin\theta - c\omega$ with $V = \omega^2/2 + 1 - \cos\theta$, one gets $dV/dt = -c\omega^2 \leq 0$. Lyapunov worked in parallel to Poincaré, by different methods, so there is no arrow between them.

Birkhoff proved Poincaré's last geometric theorem in 1913, answering Poincaré's statement of 1912. In 1931 he proved that time averages equal space averages, the ergodic theorem [26, 27, 28]. A quick check: for the rotation $x_n = n\varphi \bmod 1$ with $\varphi = (\sqrt{5} - 1)/2$, the mean of x_n over $n = 0 \ldots 10^6 - 1$ is 0.49999995, against the space average 0.5.

Kolmogorov in 1954, Moser in 1962 and Arnold in 1963 showed that most invariant tori survive small perturbations [29, 30]. Arnold gave the first detailed proof of Kolmogorov's theorem. Moser showed that analyticity is not needed.

Smale's horseshoe of 1960 codes chaotic motion exactly by infinite sequences of 0s and 1s [31, 32]. It builds on the work of Cartwright and Littlewood and of Levinson, and more loosely on Poincaré's homoclinic points.

In 1963 Lorenz found sensitive dependence on initial conditions numerically [33, 34]. A quick check: for $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, starting from $(1, 1, 1)$, fourth-order Runge–Kutta with step 0.01, discarding 50 time units and then running $T = 10{,}000$ with the tangent vector renormalised every time unit, gives a largest Lyapunov exponent of 0.905. Lorenz found this independently, not by building on Poincaré. Motter and Campbell judge that he was "in all likelihood" unfamiliar with Poincaré's work [35].

His paper cites Poincaré's 1881 paper and Birkhoff's 1927 book, but the 1890 memoir "seems to have escaped" him, according to Ghys [36].

In 2002 Tucker proved, by a computer-assisted proof, that Lorenz's attractor is a strange attractor [37, 34]. This answered the fourteenth of Smale's problems for the twenty-first century (note 4).

Fluids and particles

In 1934 Leray proved that the Navier–Stokes equations have global weak solutions [38, 39]. Whether smooth solutions can blow up is a Clay Millennium Problem. A blow-up proof was announced on 8 September 2026 [42]. On 11 September Clay called the problem "apparently settled" and its review "deliberately unhurried" [40]. On 29 October 2026 Clay's problem page still listed it as "Active" [41]. Leray's result rests on no earlier link in this chain.

In 1975 Lanford derived Boltzmann's equation from a gas of hard spheres, for short times (note 5) [43, 44, 45]. Again no earlier link of the chain is involved.

In 2024–25 Deng, Hani and Ma derived Boltzmann's equation for as long as its solution exists, and from it the fluid equations, starting from particles; the papers are preprints [44, 46, 45]. This extends Lanford's result and adds known limits from Boltzmann to the fluid equations. It does not touch the Clay question of smoothness.

Notes

1. Picard's method is settled: it appeared in 1890 for partial differential equations [47] and in 1891 for ordinary ones [48]. The years 1891–96 are those of his *Traité d'analyse*.
2. Poincaré's memoir is dated 1889 by Wikipedia, the year of the prize, and appears as Acta Mathematica 13 (1890) in Holmes's references [23]. The text uses the printed year, 1890.
3. Lyapunov defended his thesis on 12 October 1892 in the modern calendar, 30 September in the old one [49, 50]. The date "12 September" sometimes given is wrong.
4. The link from Smale to Tucker stands for Smale's fourteenth problem, which Tucker answered, not for the horseshoe. Earlier versions of this chain also drew Kepler → Newton as a proof step and d'Alembert → Fourier as a connection; the first is background only [3], and Fourier's work names only Bernoulli and Euler as precedents.
5. Lanford's paper is from 1975 [43]. His paper of 1976 is a second one [51].

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Logic and computation

One proof method holds this chain together: Cantor's diagonal argument of 1891, which belongs to the chapter on numbers and foundations. Russell used it to break Frege's system, Gödel to prove incompleteness, and Turing to prove that no algorithm decides all of logic. The formal systems that came out of this crisis are today's proof checkers. The chain has seventeen links, from Aristotle to proofs found by artificial intelligence.

Ancient and early modern logic

Around 350 BCE Aristotle showed that every valid syllogism reduces to the forms of the first figure, by conversion or by reductio [1, 2]. A quick check: a brute-force search over all 4^3 moods per figure was tested on every pattern of empty and non-empty regions of the three-term Venn diagram, with all three terms non-empty (existential import). It finds 18 valid moods in the first three figures, and 14 once the four weakened forms are removed, which is Aristotle's count. This link rests on nothing earlier.

In the third century BCE the Stoics set out five basic inferences of propositional logic [3, 4]. A quick check: all five are valid by truth table. There is no arrow from Aristotle. The Stanford Encyclopedia says that Diodorus and Philo influenced Chrysippus and worked "apparently independent of Aristotle's achievements" [3].

From 1666 into the 1680s Leibniz proposed a universal calculus of reasoning [5, 6]. It was a programme, not a theorem. According to the Stanford Encyclopedia, which quotes Harley (1867), Boole learned of Leibniz only after 1854: more than a year after *The Laws of Thought*, R. L. Ellis pointed him to Leibniz (note 1) [5]. So there is no arrow from Leibniz to Boole; the two are independent.

Logic as algebra and the foundations of arithmetic

In 1847 and 1854 Boole treated logic as algebra with the law $x^2 = x$, and derived syllogisms by elimination [7, 8, 9]. A quick check: the syllogism Barbara holds for all $16^3 = 4{,}096$ triples of subsets of a four-element set. Boole's work recasts Aristotle's syllogistic.

In 1879 Frege created the first formal logic with quantifiers, meant to derive arithmetic from logic [10, 11, 5]. It explicitly takes up Leibniz's programme and sets itself against Boole's.

Dedekind in 1888 and Peano in 1889 gave axioms for the natural numbers: zero, successor and induction [12, 13]. These rest on no earlier link of this chain.

In 1902 Russell found that the set of all sets not containing themselves contains itself if and only if it does not. Whitehead and Russell's theory of types, in *Principia Mathematica* (1910–13), blocks it [14, 11, 15]. A quick check: for every one of the $8^3 = 512$ maps from a three-element set to its subsets, the diagonal set is missed. The

paradox was derived inside Frege's system and found by applying Cantor's diagonal argument. The notation of *Principia* comes from Peano.

Completeness, incompleteness and undecidability

In 1928 Hilbert and Ackermann asked two questions: is first-order logic complete, and is it decidable [16, 17, 18]? This was a programme, not a result. Kleene's memoir confirms that the completeness question was posed in their 1928 book [19], and Wikipedia says the same of the decision problem [20].

In 1929–30 Gödel proved completeness: every valid first-order formula is provable [16, 21]. This answers Hilbert and Ackermann's first question, within their system.

In 1931 Gödel proved incompleteness. Arithmetic has true statements that cannot be proved, and no such system proves its own consistency [22, 23, 24]. A quick check: prime-power coding of (3, 1, 4, 1, 5) gives $2^3 \cdot 3^1 \cdot 5^4 \cdot 7^1 \cdot 11^5$ = 16,910,355,000, which decodes back to the same sequence. The proof uses the system of *Principia* (his system "P"), the diagonal method and the numbering of formulas. It refutes Hilbert's consistency programme as it was conceived.

In 1936 Church and Turing showed that no algorithm decides first-order logic (note 2) [25, 26]. Turing's proof is a diagonal argument on machines. It answers Hilbert and Ackermann's question and rests on Cantor's diagonal. From Gödel, Turing took the idea of encoding, but he wrote that his result is "quite different" [25].

Also in 1936 Gentzen proved that arithmetic is consistent, by induction up to the ordinal ε_0 [27, 22]. Gödel's second theorem forces any such proof to use a method beyond arithmetic.

Proofs as programs and machine proof

Curry in 1958 and Howard in 1969 showed that proofs correspond to typed programs, and that simplifying a proof is running the program [28, 29]. A quick check: the combinator expression S K K applied to 42 returns 42, as the identity should. This builds on Gentzen's natural deduction and Church's λ-calculus.

In 1965 Robinson proved that resolution with unification shows any unsatisfiable set of clauses to be unsatisfiable [30, 31, 32]. It lifts the propositional resolution rule of Davis and Putnam's procedure (1960) to first order by unification. Bonacina describes Robinson as uniting "propositional resolution (from the DP procedure) and unification" [32].

In 1971 Cook, and independently Levin, proved that satisfiability is NP-complete; whether P = NP remains open (note 3) [33, 34]. The proof encodes the run of a Turing machine as a formula.

Coq, begun in 1984, and Lean, begun in 2013, are proof checkers in which a small kernel checks every step [35, 36, 37]. They rest on proofs-as-programs and the calculus of constructions. De Moura started Lean in 2013, by his own account and that of the Simons Foundation [38, 39]. Lean 0.1 was first released in 2014 [40].

In 2024 AlphaProof and AlphaGeometry 2 scored 28 of 42 points at the International Mathematical Olympiad: 21 by AlphaProof, checked in Lean, and 7 by AlphaGeometry 2 [41, 43]. In 2025 an advanced version of Gemini scored 35 of 42, writing in natural

language [42, 44]. The scores add up: $3 \times 7 + 7 = 28$ and $5 \times 7 = 35$. Lean's kernel verified AlphaProof's three 2024 proofs; the 2025 system did not use Lean (note 4).

Notes

1. Only the Stanford Encyclopedia, quoting Harley (1867), states how and when Boole learned of Leibniz.
2. Church's theorem is reported as two different claims. MacTutor says it "extends the incompleteness proof" of Gödel's 1931 paper [45]. The later standard derivation goes through Robinson's arithmetic Q, introduced by Tarski, Mostowski and Robinson in 1953 [46, 47]; the Stanford Encyclopedia entry on the Church–Turing thesis does not describe it [17].
3. Levin published independently in 1973, in *Problemy Peredachi Informatsii* 9(3), 115–116, received on 7 June 1972 [33, 48]. Clay groups him under 1971 [34].
4. DeepMind's 2025 score was graded by IMO coordinators [42]. But IMO president Gregor Dolinar said the IMO "cannot validate the methods", including the computing power used [44]. The 2024 solutions were graded by Gowers and Myers.

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Numerical mathematics

The ancient numerical results do not form a chain. Archimedes comes before Heron, Liu Hui says he did not follow Archimedes, and the Chinese elimination method never reached Europe. The chain proper starts with Newton reading Viète in 1664. From then on each method is proved to converge, or its errors are bounded, using an earlier result. The main lines run from Newton to Kantorovich, from Euler to Runge and Kutta, from Gauss's elimination to von Neumann, Turing and Wilkinson, and from Newton's method into the interior-point methods of linear programming. The chain has twenty-one links, from the tablet YBC 7289 to 2022.

Ancient approximations without a chain

The Babylonian tablet YBC 7289, from about 1800–1600 BCE, gives $\sqrt{2} \approx 1;24,51,10$ in base 60 [1, 2]. No proof or method is recorded. A quick check: the value is 1.41421296, an error of -6.0×10^{-7}.

Around 250 BCE Archimedes proved $3\ 10/71 < \pi < 3\ 1/7$ [3, 4, 5, 6]. He doubled the sides of the inscribed and circumscribed regular hexagons, $6 \to 12 \to 24 \to 48 \to 96$. A quick check: $96 \sin(\pi/96) = 3.141032$ and $96 \tan(\pi/96) = 3.142715$, which lie inside his bounds. The proof uses the bounds $265/153 < \sqrt{3} < 1351/780$ [4, 5]; he does not say how he found them (note 1).

Around 60 CE Heron gave the rule $x \leftarrow (x + a/x)/2$, whose error shrinks quadratically [7, 8]. A quick check: from 3/2 the iterates 17/12, 577/408 and 665857/470832 are off by 2.5×10^{-3}, 2.1×10^{-6} and 1.6×10^{-12}. Whether the Babylonians already used this method is not known, because how YBC 7289 was computed is not known (note 2) [1, 9]. Gander and co-authors observe that the second iterate from 3/2, namely $577/408 = 1;24,51,10,35\ldots$, gives the tablet's value when cut off after three places [9].

The Chinese *Nine Chapters*, compiled between about 200 BCE and 50 CE, solves linear systems by *fangcheng*: elimination on an array of coefficients by column operations [10, 11]. A quick check: problem 1 ($3x + 2y + z = 39$, $2x + 3y + z = 34$, $x + 2y + 3z = 26$) gives (9.25, 4.25, 2.75). This rests on no earlier link.

In 263 Liu Hui computed π from inscribed polygons only, with a correction based on area [12, 13, 14]. A quick check: for the inscribed 3072-gon in the unit circle, the half-perimeter $3072 \sin(\pi/3072)$ is 3.1415921 and the area $1536 \sin(2\pi/3072)$ is 3.1415905. There is no arrow from Archimedes. Liu Hui used areas of inscribed polygons, not Archimedes' inscribed and circumscribed perimeters.

Qin Jiushao in 1247, and Ruffini and Horner between 1809 and 1830, found numerical roots of polynomial equations [15, 16]. Ruffini's synthetic division is equivalent to Horner's method. The method belongs to the Chinese tradition; Ruffini and Horner rediscovered it independently (note 3).

Newton's method and its convergence

Newton in 1669, Raphson in 1690 and Simpson in 1740 developed the method of linearising and repeating [17, 18]. Raphson iterated on the original polynomial, and Simpson added derivatives and extended it to systems. A quick check: for $x^3 - 2x - 5$ from $x_0 = 2$, computed with 30 digits, the iterates are 2.1, 2.0945681 and 2.0945515, and the error is squared at each step. The method rests on Viète's numerical method, which Newton studied in 1664 [18].

Kantorovich carried Newton's method into Banach spaces [18, 19]. In 1939 he proved local linear convergence, and in 1948–49 local quadratic convergence under conditions on the derivative. This rests on Newton's iteration. No convergence proof for Newton's method before Kantorovich has been found (note 4).

Differential equations, interpolation and quadrature

In 1768 Euler introduced the step $y_{n+1} = y_n + h\, f(x_n, y_n)$ [20, 21, 22]. Cauchy later proved that the Euler polygons converge to the solution; according to Hairer and Wanner he did so in lecture notes of about 1824 (note 5) [20, 22]. A quick check: for $y' = y$, $y(0) = 1$, the errors at $x = 1$ for $h = 0.1$, 0.05, 0.025 and 0.0125 are 0.1245, 0.0650, 0.0332 and 0.0168. Halving h about halves the error, as a first-order method should.

Runge in 1895, Heun in 1900 and Kutta in 1901 used several stages per step, matched to the Taylor series [23, 24]. A quick check: for $y' = y$, $y(0) = 1$, the error of the classical fourth-order method at $x = 1$ falls from 2.08×10^{-6} to 1.36×10^{-7} when h halves from 0.1 to 0.05. That is a factor of $15.35 \approx 2^4$. In Butcher's words, Runge "extended the approximation method of Euler" [23].

In 1814 Gauss showed that n well-chosen nodes integrate polynomials exactly up to degree $2n - 1$; in 1826 Jacobi reproved it by orthogonality (note 6) [27, 28, 29]. A quick check: on $[-1, 1]$ the three-point rule is exact for $x^5 + x^4$, giving 0.4, but gives 0.24 instead of 2/7 for x^6. Gauss started from the Newton–Cotes rules. His memoir first treats the equally spaced rules of Newton and Cotes, then lets the nodes move freely [27, 29].

In 1901 Runge showed that interpolants of an analytic function at equally spaced nodes can diverge, while Chebyshev nodes converge [30, 31]. A quick check: for $1/(1 + 25x^2)$ on $[-1, 1]$ at degree 20 (21 nodes), the maximum errors on 10,001 test points are 59.8 with equally spaced nodes and 0.015 with Chebyshev nodes of the first kind. This is a counterexample and rests on no earlier link.

Linear algebra: elimination, its errors and its successors

In 1809–10 Gauss solved least-squares problems through the normal equations, by what he called "common elimination" [25, 26]. A quick check: for a random 100×5 system (numpy's default generator, seed 0), the normal equations and numpy's least-squares solver agree to 1.4×10^{-16}. Gauss used elimination as it had been taught since Newton. There was no transmission from China [25].

In 1947 von Neumann and Goldstine bounded the rounding errors of elimination [32, 33]. They described elimination as triangular factoring and introduced a condition

number. Their work rests on Gauss's elimination for symmetric positive definite matrices.

In 1948 Turing described elimination as LU factorisation and coined the term "condition number" [32, 34]. He followed the 1947 paper, which had triangular (LDU) factoring first [32, 34]. Grcar alone calls the 1947 paper Turing's inspiration.

Between 1954 and 1965 Wilkinson developed backward error analysis: elimination with pivoting solves a nearby system exactly (note 7) [35, 34, 33]. A quick check: for random 200 × 200 systems with standard normal entries (seeds 0–4), solved by LU with partial pivoting, the backward error $\|b - A\hat{x}\|_2/(\|A\|_F \|\hat{x}\|_2)$ is 1.1–1.4 × 10^{-16}, near machine precision. The analysis rests on von Neumann and Goldstine's rounding-error axioms [34, 33]. Turing counts only as an influence: Wilkinson was his assistant, but no source says the proof uses Turing's paper.

In 1952 Hestenes and Stiefel proved that conjugate gradients finish in at most n steps in exact arithmetic [36, 37]. A quick check: for $A = MM^T + 50I$ with M a 50 × 50 standard normal matrix (seeds 0–4), the method reached a relative residual of 10^{-10} in 21–22 steps. Gauss's elimination is a special case of their family of conjugate-direction methods.

In 1943 Courant applied the Rayleigh–Ritz method with trial functions on triangles; engineers developed it into the finite element method between 1956 and 1973 [38, 39, 40]. This rests on Ritz's method. Liu, Li and Park write that Courant "systematically used the Rayleigh Ritz method" [38]. Gander and Wanner write that "the value of Ritz' invention was only recognized much later by Courant", who gave the first finite element calculation in an address to the American Mathematical Society [40].

In 1965 Cooley and Tukey published the fast Fourier transform, which needs about N log N operations [41, 42]. A quick check: for N = 1024 complex standard normal inputs (seeds 0–2), numpy's FFT matches the direct transform to 2.3–2.4 × 10^{-11}, about 2 × 10^{-13} in relative terms. The saving is $N^2/(N \log_2 N) = 102.4$ times fewer operations. The paper itself cites Good and Yates as its basis [41]. Gauss had an equivalent algorithm around 1805, but it was rediscovered, not used: the two are independent [42].

Optimisation

Dantzig's simplex method of 1947 and Karmarkar's interior-point method of 1984 solve linear programmes; Karmarkar's method is equivalent to a logarithmic barrier method [43, 44, 45]. A quick check: maximising 3x + 5y subject to $x \le 4$, $2y \le 12$, $3x + 2y \le 18$ and $x, y \ge 0$ gives (2, 6) with value 36, with both a simplex and an interior-point solver. The interior-point methods apply Newton's method to the barrier conditions and use Frisch's barrier.

In 2022 Chen, Kyng, Liu, Peng, Probst Gutenberg and Sachdeva computed maximum flows in almost-linear time [46, 47]. The paper describes "a new potential reduction IPM … inspired by" Karmarkar [46]. The popular account in Quanta does not name Karmarkar [47].

Notes

1. Only Wikipedia states that Archimedes does not say how he found his bounds for √3 [4].
2. MacTutor's page on Babylonian mathematics calls Babylonian use of Heron's method "no more than a fairly remote possibility" [1], and Wikipedia says "it is not known exactly how" [8]. Gander and co-authors agree that the method behind YBC 7289 is unknown [9]. Only MacTutor's page on Heron says the Babylonians knew it [7].
3. Horner's date is not settled by a primary source. The 1819 paper is usually cited, but its method differs from "Horner's method" [15, 16]. MacTutor locates the method in Horner's 1830 paper, and Wikipedia, following Fuller, gives priority to Holdred (1820).
4. No source was found for a convergence proof of Newton's method before Kantorovich.
5. Cauchy's convergence proof for Euler's method is confirmed [20, 22], and Moigno's course based on Cauchy's lectures includes an existence proof by Euler's method [48]. The date of about 1824 for Cauchy's notes rests on Hairer and Wanner alone. The record of the 1981 edition of the course, with Gilain's introduction, gives no date [49].
6. Gauss's quadrature memoir is dated 1814 here, while Sanz-Serna cites the printed memoir as 1815 [29].
7. Whom Wilkinson credited with the first backward error analysis is unclear. Higham reports that he credited both von Neumann and Goldstine (1947) and Givens [35]. Dopico reports Givens (1954) [34]. They differ on which he called first.

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Unifying structures

The first half of this chain is a chain of definitions, not proofs. From Klein to Bourbaki and the first categories, each step reorganised mathematics around structures. The sources describe succession and influence rather than one proof using another. The exception is Hilbert, whose consistency proof of 1899 uses coordinates in the manner of Descartes as its model. From Grothendieck on, the arrows are proofs again: his cohomology is what Deligne used, and Deligne's theorem is used by Ngô and Scholze. A second line runs from Langlands's conjectures through Wiles to 2025. The chain has eighteen links, from Descartes to 2025.

Coordinates, surfaces and groups

In 1637 Descartes turned geometric problems into equations [1, 2]. He solved Pappus's problem and classed curves by their degree. A quick check: the circle $x^2 + y^2 = 25$ meets the line $y = x + 1$ at $(-4, -3)$ and $(3, 4)$. This link rests on nothing earlier in the chain.

In 1851 and 1857 Riemann showed that topology governs complex function theory, and introduced the genus of a surface [3, 4]. His work rests on Dirichlet's principle, which Hilbert made rigorous in 1901.

In 1872 Klein defined a geometry as the study of the invariants of a group of transformations, in his Erlangen Program [5, 6]. A quick check: with $(z_1, z_2; z_3, z_4) = (z_1 - z_3)(z_2 - z_4)/((z_2 - z_3)(z_1 - z_4))$, the cross-ratio of $(1, 2, 5, -3)$ is $5/3$ both before and after the Möbius map $z \mapsto (2z + 1)/(z + 4)$. The distance between 1 and 2, by contrast, changes from 1 to 7/30. Cayley's cross-ratio metric was an influence [5]. Riemann's geometric approach shaped Klein's function theory, not the Erlangen Program itself [5, 6] (note 1).

Axioms and algebraic structures

In 1899 Hilbert gave axioms for Euclidean geometry and showed that they are consistent if real analysis is [7, 8, 9]. He read points as pairs of real numbers. This builds on coordinates in Descartes' sense, although the sources describe the model only as pairs of reals and do not name Descartes.

In 1921 Noether proved primary decomposition in every ring with the ascending chain condition (note 2) [10, 11, 12, 13]. This generalises Lasker's theorem. Hilbert's axiomatic approach was an influence.

In 1930–31 van der Waerden published *Moderne Algebra*, one of the first structural textbooks of algebra [14, 15]. It rests on Noether's and Artin's lectures.

From 1939 Bourbaki organised mathematics by algebraic, order and topological structures [16, 17]. Corry calls this "a step beyond van der Waerden" [16].

In 1945 Eilenberg and Mac Lane introduced categories, functors and natural transformations [18, 16]. This grew out of their 1942 work on group extensions and

homology. There is no arrow from Bourbaki: the two developed separately, and Corry says that category theory overtook Bourbaki's structures [16].

Grothendieck, the Weil conjectures and Deligne

In 1957, in the paper known as Tôhoku, Grothendieck introduced abelian categories and derived functors [18, 19]. Sheaves of modules form an abelian category, so homological algebra applies to geometry. Here categories are used intrinsically.

In 1949 Weil stated conjectures for the zeta functions of varieties over finite fields, and proved them for curves [20, 21]. A quick check: for the curve of conductor 11, $y^2 + y = x^3 - x^2 - 10x - 20$, brute-force point counts give $|a_p| \le 2\sqrt{p}$ for all 14 primes up to 43. The prime 11 is the bad prime, with $a_{11} = 1$. Counting over the field with nine elements gives $\#E(\mathbb{F}_9) = 15 = 9 + 1 - (a_3^2 - 2 \cdot 3)$, as predicted. Weil's work rests on Hasse's proof for genus 1.

From 1960 to 1965 Grothendieck developed schemes and étale cohomology [21, 22, 23, 24, 25]. He proved the rationality of the zeta function by the Lefschetz trace formula; Dwork had found a p-adic proof in 1959. This rests on the homological algebra of Tôhoku, which Jardine calls "the first decisive step towards proving the Weil conjectures" [24]. Étale cohomology is defined by derived functors [25].

In 1974 Deligne proved the Riemann hypothesis over finite fields (note 3) [26, 22]. He wrote: "Nous ferons un usage essentiel des résultats de Grothendieck" (we shall make essential use of Grothendieck's results) [26]. The proof also uses Rankin's tensor-power trick.

The Langlands programme

In 1967 Langlands stated his conjectures: functoriality, and a reciprocity between Galois representations and automorphic forms [27, 28]. This is a conjecture, not a proof.

In 1995 Wiles, with Taylor and Wiles, proved that semistable elliptic curves over $\mathbb{Q}$ are modular, so that Fermat's Last Theorem holds [29, 30]. A quick check: for the curve 11a, $y^2 + y = x^3 - x^2 - 10x - 20$, the coefficients of $q\prod(1 - q^n)^2(1 - q^{11n})^2$ match $a_p = p + 1 - \#E(\mathbb{F}_p)$ for all 14 primes up to 43. The Langlands–Tunnell theorem supplies the starting point, modularity mod 3. Wiles used that later theorem, not the 1967 letter itself (note 4).

From 2008 to 2010 Ngô proved the fundamental lemma, which is needed to stabilise the trace formula [31, 32]. He worked within Langlands's programme. The proof applies Deligne's purity and the decomposition theorem to the Hitchin fibration.

From 2011 to 2021 Scholze developed perfectoid spaces, first in arXiv 1111.4914 of 21 November 2011, published in Publications Mathématiques de l'IHÉS 116 (2012), 245–313 [33, 34, 35, 36]. They allow tilting between characteristic 0 and characteristic p. With them he proved the weight–monodromy conjecture for complete intersections. With Fargues he gave a geometrization of the local Langlands correspondence, posted in 2021 and published in *Astérisque* 466 in 2026 [37, 38]. This work rests on Deligne's theorem in characteristic p, on Fontaine and Wintenberger, and on Faltings.

In 2024 Gaitsgory, Raskin and seven others proved the geometric Langlands conjecture, in its categorical and unramified form, in five papers (note 5) [39, 40]. The proof rests on the Beilinson–Drinfeld construction and on the formulation of Arinkin and Gaitsgory of 2012. Geometric Langlands is an analogue of the arithmetic programme, and there is no arrow to it: the two results are parallel.

In 2025 Boxer, Calegari, Gee and Pilloni proved that a positive proportion of abelian surfaces over $\mathbb{Q}$ are modular [41, 42]. On 29 October 2026 this was still a preprint: arXiv 2502.20645 has only its first version, of 28 February 2025, and no journal reference. The proof uses the Taylor–Wiles method, with a "2–3 switch". Only the authors' own texts describe this; the survey cited is by Gee, a co-author [42].

Notes

1. Klein's book on Riemann's theory dates from 1882 [6, 43]; only the MacTutor page on Riemann gives 1892. The influences of Riemann on Klein's function theory and of Cayley on Klein are not proof steps.
2. Noether's paper was received on 16 October 1920 and published in 1921 [12].
3. Deligne's paper was received on 20 September 1973 and published in 1974 [26].
4. Wiles used the later Langlands–Tunnell theorem, and Ngô proved the fundamental lemma within Langlands's programme. Neither proof uses the 1967 letter itself, so these links are influence rather than proof dependency.
5. Gaitsgory's outline dates from 2013 [44]; *Nature*'s date of 2015 refers to its publication in *Astérisque* 370. The length of the proof is reported differently: "more than 800 pages" in Quanta [39] and "almost 1,000 pages" in *Nature* [40]. None of the five arXiv pages, nor Gaitsgory's project page, states a page count. On 29 October 2026 the arXiv pages showed: I, latest version v3 of 30 September 2025 [45]; II, v3 of 11 September 2024 [46]; III, only v1 of 11 September 2024 [47]; IV, only v1 of 13 September 2024 [48]; V, v3 of 16 January 2026 [49]. None has a journal reference, and neither Gaitsgory's project page [50] nor nLab [51] names a journal. *Nature* (17 July 2025) says the papers "have all been submitted to journals" and that peer review will take years [40]. So the proof stands with refereeing in progress.

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Order and lattices

No ancient roots of this field can be documented. The chain starts with Boole's algebra of logic and turns on a gap: Peirce's claim of 1880 that distributivity follows from the lattice laws, refuted by Schröder in 1890 and Dedekind in 1897. After Birkhoff rebuilt the subject in the 1930s and Stone treated the Boolean case, the chain splits into two lines. One is set-theoretic: maximal principles, extensions of orders and fixed points. The other concerns finite partially ordered sets: Dilworth's theorem, the 1/3–2/3 conjecture and the Dedekind numbers. In 2026 the 1/3–2/3 conjecture is still open; a July preprint verifies it through 14 elements, and a September preprint claims the related Kahn–Saks conjecture. Both are unrefereed. Birkhoff's representation of finite distributive lattices and Stone's representation of Boolean algebras are independent: Stone came from spectral theory. The chain has twenty-four links, from Boole to Aires.

From the algebra of logic to Dedekind's lattices

In 1847 and 1854 Boole treated logic as an algebra of classes, with idempotent multiplication xx = x, in *The Mathematical Analysis of Logic* and *The Laws of Thought* [1, 2, 3]. This rests on nothing earlier in the chain.

In 1880 Peirce, building on Boole's algebra of logic, claimed that the distributive law follows from the lattice axioms [1, 4, 5]. He called the proof "too tedious to include", and in 1885 he admitted that he could not prove it. This is a gap: no proof was given. His letter of 1903, printed by Huntington in 1904 (Transactions of the AMS 5, 288–309), offered a proof that uses complements, so it does not cover lattices in general (note 1).

In 1890 Schröder showed that distributivity is independent of the other laws [1, 6, 7]. In an appendix to the first volume of his *Vorlesungen* he gave three concrete models that satisfy the other laws but are not distributive. This refutes Peirce's claim by counterexample.

In 1897 Dedekind, in his paper on *Dualgruppen*, defined lattices by meet and join (note 2) [1, 8]. He showed that lattices of submodules satisfy the modular identity, which equals its own dual, and that a five-element lattice is not distributive. A quick check: in the lattice M3 of the three lines of $GF(2)^2$, $a \wedge (b \vee c)$ has 2 vectors while $(a \wedge b) \vee (a \wedge c)$ has 1. The five-element example refutes Peirce's claim.

In the same paper of 1897 Dedekind counted the monotone Boolean functions, which are the elements of the free distributive lattice [9, 10]. He found M(0), …, M(4) = 2, 3, 6, 20, 168. A quick check: brute force gives [2, 3, 6, 20, 168], and further M(5) = 7581; the free distributive lattice on three generators has $20 - 2 = 18$ elements. This counting problem was posed on the basis of his free distributive lattices.

In 1900, in "Über die von drei Moduln erzeugte Dualgruppe" (Mathematische Annalen), Dedekind showed that the free modular lattice on three generators has 28 elements, with no top or bottom added [8, 11, 7]. A quick check: in the product of the subspace lattice of $GF(2)^2$ (5 subspaces) over all 125 triples, the three coordinate

generators generate a sublattice of 28 elements, so |FM(3)| ≥ 28. The count uses the modular *Dualgruppe* of 1897; no source says it uses that paper beyond its definitions.

Maximal principles and fixed points

In 1909 Hausdorff gave the first explicit maximal principle and derived it from the axiom of choice; his *Grundzüge* of 1914 contains several (note 3) [12, 13, 14, 15]. In today's form it says that every chain lies in a maximal chain [14, 15]. It rests on nothing earlier in the chain.

In 1927–28 Knaster, with Tarski, proved that a monotone map on the lattice of subsets of a set has a fixed point: take the join of all x with x ≤ f(x) (note 4) [16, 17]. A quick check: with f(X) = {0} ∪ {x + 3 : x ∈ X, x + 3 ≤ 9} on the subsets of {0, …, 9}, iterating f from ∅ reaches the least fixed point {0, 3, 6, 9}.

In 1930 Szpilrajn proved that every partial order extends to a linear order (Fundamenta Mathematicae 16, 386–389) [18, 19, 20]. A maximal extension must be total, since an incomparable pair could still be added. The proof uses a maximal principle. According to arXiv 1708.04711, the original proof takes a maximal extension of an enlarged order: "The original proof of the theorem splits into two steps" [19]. Wikipedia gives only the standard proof by Zorn's lemma [18].

Birkhoff, Stone and the general theory

In 1933 Birkhoff introduced lattices as a general concept, in "On the combination of subalgebras" (Proceedings of the Cambridge Philosophical Society 29, 441–464) [21, 7, 22]. Ore worked in parallel and called lattices "structures" [7]. That the subalgebras of any algebra form a complete (algebraic) lattice is standard [22]; whether it is a theorem of this paper is not shown by the sources. Dedekind's work of 1897 was an influence only.

In 1936 Stone proved that every Boolean algebra is the field of clopen sets of its Stone space (Transactions of the AMS 40) [23, 13, 24]. The points are ultrafilters, and b ↦ {ultrafilters containing b}. The theorem is equivalent to the Boolean prime ideal theorem, a choice principle. Hausdorff's principle is not named as its basis (note 5). It does not depend on Birkhoff's paper of 1937, which appeared a year later.

In 1937, in "Rings of sets" (Duke Mathematical Journal 3), Birkhoff proved that every finite distributive lattice is the lattice of down-sets of its join-irreducible elements, which are join-prime [25, 26, 27]. A quick check: the free distributive lattice on three generators has 18 elements. The general lattice framework of his 1933 paper is background, not a proof step. Neither Wikipedia nor Rota says that the 1937 proof uses the 1933 paper, and Rota dates the result to "the early thirties" (note 6) [26].

In 1940 Birkhoff published the first edition of *Lattice Theory* (AMS Colloquium Publications 25, 6 + 155 pages), reviewed by L. R. Wilcox in the Bulletin of the AMS 47, 194–196 [28, 29]. It is a monograph, with no single new theorem. No source consulted says that it fixed the shape of the field.

Counting and finite posets

The Dedekind numbers were extended by ever more clever enumeration of the free distributive lattice, building on Dedekind’s problem of 1897 [9, 30]. Church found M(5) = 7581 in 1940 (Duke Mathematical Journal 6), printed as 7579 according to Mathematical Reviews. Ward found M(6) = 7,828,354 in 1946, Church M(7) in 1965, and Wiedemann M(8) in 1991 (Order 8). A quick check: M(5) recomputes as 7581.

In 1950 Dilworth proved that the maximum size of an antichain equals the minimum number of chains needed to partition the poset [31, 32, 33]. The textbook proof by induction is due to Perles (1963) or Galvin (1994); Dilworth’s own method was not consulted. Fulkerson gave a proof via bipartite matching (Proceedings of the AMS 7, 701–702, 1956), which Wikipedia ties to Kőnig’s theorem; Georgia Tech dates it 1954 [31, 32, 33]. A quick check on a random 11-element poset (each pair i < j related with probability 0.3, then transitive closure): the maximum antichain has 5 elements, and 11 minus a maximum matching is also 5. Dilworth’s theorem rests on no earlier link.

In 1955 Tarski proved that on any complete lattice the fixed points of a monotone map form a complete lattice [16, 17]. This generalises the Knaster–Tarski theorem. In the same year Davis proved the converse: a lattice in which every monotone map has a fixed point is complete.

Kislitsyn in 1968, Fredman in 1976 and Linial in 1984 posed the 1/3–2/3 conjecture (note 7) [34, 35]. It says that every finite poset that is not a chain has a pair x, y with 1/3 ≤ P(x < y) ≤ 2/3, where the probability is over random linear extensions. It arose from sorting under partial information. A quick check: for the poset {a < b, c}, P(a < c) = 2/3, so the balance δ is exactly 1/3, the extremal case.

In 1971 Mirsky proved the dual of Dilworth’s theorem: the length of the longest chain equals the minimum number of antichains in a partition [31, 33]. The proof layers the elements by the longest chain ending at each. A quick check: the same random poset has a longest chain of 4 and 4 layers. Mirsky’s theorem was proved independently by this layering, not from Dilworth’s theorem.

In 1984 Kahn and Saks gave the first uniform bound, δ(P) > 3/11 = 0.2727… (Order 1, 113–126) [34, 36, 37, 38]. It comes from log-concavity inequalities proved with the Alexandrov–Fenchel inequalities, like Stanley’s inequality. This is a partial result towards the 1/3–2/3 conjecture.

In 1995 Brightwell, Felsner and Trotter improved the bound to δ(P) ≥ (5 − √5)/10 by sharpening the same approach (note 8) [36, 34, 38, 39]. A quick check: (5 − √5)/10 = 0.276393…, above 3/11 and below 1/3. In their own words, the result “requires all the machinery developed by Kahn and Saks” [36].

In 2023 Jäkel, and independently Van Hirtum and co-authors, found M(9) = 286386577668298411128469151667598498812366 [10, 40, 41, 30, 9]. Jäkel used symmetries and matrix multiplication (arXiv, 3 April 2023). He presented the result at ICFCA 2023 in Kassel, according to TU Dresden, and it appeared in the Journal of Computational Algebra 6–7 (2023), 100006, according to OEIS and Wikipedia. Van Hirtum and co-authors counted with an FPGA supercomputer; their result was committed on 8 March 2023, according to arXiv 2304.03039 and TU Dresden (note 9). A quick check: the number has 42 digits and matches OEIS. It was not recomputed, since that needs a supercomputer. Both counts rest on Dedekind’s counting problem.

Recent work on balance

In 2025 Aires and Kahn proved that δ(P) → 1/2 when the width is Ω(n), and δ(P) ≥ 1/e − o(1) when the width is ω(√n) (arXiv 2509.11549, September 2025) [42, 43, 44]. A second paper (arXiv 2510.26134, October 2025) derives the Kahn–Saks conclusion from a large variance of an element's position in a random linear extension. This proves the Kahn–Saks conjecture for width Ω(n). Both papers are arXiv preprints with no journal reference as of 29 October 2026. Aires spoke on the work at the Institute for Advanced Study on 22 September 2025.

On 27 July 2026 Gupta posted exact balance data for all 1,338,193,159,771 unlabelled 14-element posets [45, 35]. The 1/3–2/3 conjecture and the Gold Partition Conjecture hold through 14 elements, and the least balance constant above 1/3 is 37/106. The computation is exhaustive and exact, and the extremal cases were rechecked independently. A quick check: 37/106 = 0.349057 and 254/725 = 0.350345, both above 1/3. The result verifies the 1/3–2/3 conjecture for a finite range, by way of Peczarski's Gold Partition Conjecture. It was an unrefereed preprint as of 28 October 2026.

On 25 September 2026 Aires claimed a proof of the Kahn–Saks conjecture: δ(P) → 1/2 as the width tends to infinity [46, 47]. Large width forces either a nearly uniform order on k elements or a nearly fixed order with one element inserted uniformly. The paper states that it builds on the two Aires–Kahn papers of 2025, which had obtained δ(P) → 1/2 when the width is Ω(n) or |min(P)| = ω(log n) [46]. It was an unrefereed preprint as of 28 October 2026 (note 10).

Notes

1. Only the Stanford Encyclopedia describes Peirce's 1903 proof as using complements; Huntington's text itself was not consulted.
2. Wikipedia gives 1894 for Dedekind noticing the modular law. The Stanford Encyclopedia uses 1897, the Encyclopedia of Mathematics cites 1900, and Rota gives no year. The year 1897 is taken here as the first publication.
3. Hausdorff first stated his maximal principle, and derived it from the axiom of choice, in 1909, then restated it in the *Grundzüge* of 1914 [12, 13]. Wikipedia's 1914 is the restatement.
4. Tarski's footnote of 1955 says 1927, while his bibliography and Wikipedia say 1928. The theorem was probably found in 1927 and published in 1928.
5. Whether Stone's proof uses Hausdorff's principle is open: Stone's 1936 paper was not read for its choice principle.
6. Rota dates Birkhoff's representation of finite distributive lattices to the "early thirties", Wikipedia to 1937.
7. Who posed the 1/3–2/3 conjecture after Kislitsyn is not settled. Chen and Brightwell–Felsner–Trotter name Fredman, Chen with 1976 [34, 36]. Wikipedia notes that Fredman's 1976 paper is not known to mention the conjecture, and dates Linial's posing to 1984 [35]. Brightwell–Felsner–Trotter give 1969 for Kislitsyn; Chen and Wikipedia give 1968.
8. The pages of the Brightwell–Felsner–Trotter paper are Order 12 (1995), 327–349 in arXiv 1302.5967 [38], but 321–335 in Felsner's publication list [39].
9. That Van Hirtum and co-authors' count appeared in ACM Transactions on Reconfigurable Technology and Systems in 2024 is stated only by Wikipedia [9].

10. In September 2025 the IAS abstract said the authors' results were "still far from" both conjectures [44], and a Harvard talk announcement of May 2026 also said they were "still far from proving" the conjecture [47]. The September 2026 claim rests so far only on the author's own preprint. Von Neumann's continuous geometry and the quantum logic of Birkhoff and von Neumann (1936) are not entered as links in this chain.

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Universal algebra

In this field the name came before the results. Whitehead's *Treatise* of 1898 has no general theorems, and the proven chain starts with Birkhoff's general algebras of 1933 and his HSP theorem of 1935. From there three lines run out. One concerns clones: Post in 1941, Janov and Mučnik in 1959, Rosenberg in 1970. A second concerns Mal'cev conditions: Mal'cev in 1954 and Jónsson in 1967. A third concerns decidability: Tarski's finite basis problem, shown undecidable by McKenzie. The lines meet in the constraint satisfaction problem. The dichotomy conjecture of Feder and Vardi was proved by Bulatov and by Zhuk in 2017, and Zhuk's simplified proof of 2024 is still a preprint in October 2026. Post's countable lattice and the continuum of clones on three elements are independent results, and so are the two proofs of 2017. The chain has seventeen links, from Whitehead to Zhuk's paper of 2024.

Foundations: Whitehead and Birkhoff

In 1898 Whitehead published *A Treatise on Universal Algebra* [1, 2, 3]. According to Wikipedia it was a programme with "no results of a general nature" (note 1) [1]. Wikipedia says the term already had essentially its modern sense, while MacTutor says it meant something different to him [1, 2]. He took the word from Sylvester [1, 2]; Wikipedia adds that he credited Hamilton and De Morgan [1].

In 1933, in "On the combination of subalgebras" (Proceedings of the Cambridge Philosophical Society 29), Birkhoff gave a general definition of an algebra, one that allowed infinitary operations [3, 1, 4, 5]. Birkhoff and Ore began publishing on universal algebras in the early 1930s [1]. That the subalgebras of an algebra form a complete lattice is standard [5], but the sources do not show that it is a theorem of this paper. Whitehead's book was an influence only; the definition builds on complete lattices, from the chapter on order and lattices.

In 1935, in "On the structure of abstract algebras" (Proceedings of the Cambridge Philosophical Society 31, 433–454), Birkhoff proved the HSP theorem [6, 7, 8]. A class of algebras is equational if and only if it is closed under homomorphic images, subalgebras and products. Free algebras exist in every non-trivial variety, and every member is an image of one. A quick check: closing x = (0, 0, 1, 1) and y = (0, 1, 0, 1) under ∧, ∨ and ¬ in $\{0, 1\}^4$ gives the free Boolean algebra on two generators, with 16 = 2^(2²) elements. The theorem builds on his general definition of an algebra of 1933, although no source says the 1935 proof uses that paper.

Clones

In 1941 Post gave a complete description of all clones on {0, 1} [9, 10, 11]. There are countably many, with five maximal clones: P0 and P1 (functions preserving 0 or 1), M (monotone), D (self-dual) and A (affine). The criterion for functional completeness follows. Pelletier and Martin call Post's method "algebraic in nature": it describes the classes that a set of generators produces [10]. A quick check: NAND lies in none of

the five maximal clones, so it is complete on its own; of the 16 binary functions exactly two lie in none, NAND and NOR. Post's work rests on no earlier link.

In 1959 Janov and Mučnik proved that there are continuum many clones on any finite set with three or more elements [19, 11]. Their method is not described in the sources consulted. There is no dependency on Post: the result stands in contrast to his countable lattice.

In 1970 Rosenberg classified the maximal clones on every finite set into six relational families, generalising Post's five (note 2) [24, 25, 11, 26]. The two-element case is Post's, so this generalises Post's result of 1941.

Mal'cev conditions and decidability

In 1954 Mal'cev proved that a variety is congruence-permutable if and only if it has a term p with p(x, x, z) = z and p(x, z, z) = x [12, 13, 14, 15, 16]. Equivalently, its free algebra is congruence-permutable; arXiv 2107.08456 states this with the free algebra on two generators [15]. The paper appeared in Matematicheskii Sbornik 35(77), 3–20 (note 3) [14]. A quick check: $p = x \cdot y^{-1} \cdot z$ on the symmetric group S3 satisfies both identities for all 36 pairs. None of the 20 monotone ternary operations on the two-element chain is a Mal'cev term, so lattices are not congruence-permutable. The theorem uses Birkhoff's free algebras of 1935, which appear in the equivalent conditions, though no source says that the proof cites Birkhoff.

In 1955 Mal'cev introduced the algebras now named after him: anticommutative algebras satisfying the Malcev identity [17, 18]. MacTutor describes them as generalisations of Lie algebras [18]. According to Wikipedia they are the tangent algebras of analytic Moufang loops [17]. This link crosses into the chapter on non-associative algebras. A quick check: for 200 random triples of imaginary octonions (Cayley–Dickson construction, integer coordinates in $[-3, 3]$) under the commutator, $(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$ held in all 200, while the Jacobi identity failed in all 200. No source ties Malcev algebras to his universal-algebra paper of 1954, so the two are independent.

In the early 1960s Tarski posed the finite basis problem: is it decidable whether a finite algebra is finitely based [20, 21]? This is a problem, not a proof. In Barto's words, "In early 1960s A. Tarski posed" it [21].

In 1967, in "Algebras whose congruence lattices are distributive" (Mathematica Scandinavica 21, 110–121), Jónsson proved that a variety is congruence-distributive if and only if it has a sequence of Jónsson terms [22, 23, 16, 21]. Jónsson's lemma locates the subdirectly irreducible members of HSP(K) in $\mathrm{HSP_u}(K)$, the class built with ultraproducts [21]; that the lemma is in this paper is stated only by Barto's notes. A quick check: the lattice majority term $m = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$ gives m(x, x, y) = m(x, y, x) = m(y, x, x) = x on the two-element chain, on M3 and on N5. This is the case n = 2 of a Jónsson sequence. Jónsson's result is an analogous Mal'cev condition, not a step that uses Mal'cev's proof. Wikipedia calls Jónsson terms "one of the earliest discovered" Mal'cev conditions [22].

In 1996 McKenzie showed that Tarski's finite basis problem is undecidable, in a series of three papers [21, 20, 32]. His algebra A(T), built from a Turing machine T, has a finite residual bound if and only if T halts, and is not finitely based if T does not halt [20, 32]. His own route to undecidability was not consulted. This answers Tarski's

problem negatively. The companion paper is in the International Journal of Algebra and Computation 6 (1996), pp. 29–48 [32] (note 4).

In 1997 Willard proved that McKenzie's A(T) is finitely based if T halts (Transactions of the AMS 349, 2755–2774; received 18 October 1995) [20, 33, 32]. This gives a shorter proof of undecidability and removes A(T) as a route to Jónsson's problem. It streamlines McKenzie's work.

The constraint satisfaction problem

In 1978 Schaefer proved that every Boolean constraint satisfaction problem is either in P or NP-complete [27, 28, 29, 24]. It is tractable when the relations are 0-valid, 1-valid, Horn, anti-Horn, affine or bijunctive. Schaefer's own method was not consulted. A quick check: the majority operation preserves all 15 non-empty binary Boolean relations, which is why 2-SAT is tractable, while the 3-clause relation, with 7 tuples, is not closed under majority. Later re-proofs by Böhler, Creignou, Reith and Vollmer, and by Allender, Bauland, Immerman, Schnoor and Vollmer, "rely heavily" on clones and use Post's classification of all Boolean clones [27, 29]. Whether Schaefer himself used Post's lattice is not known (note 5).

Feder and Vardi conjectured that every constraint satisfaction problem on a finite domain is in P or NP-complete [30, 24, 31]. Barto and Kozik date the conjecture to 1993; the journal paper appeared in the SIAM Journal on Computing in 1998 [31, 30]. Barto and Kozik name Schaefer's theorem and the Hell–Nešetřil theorem as its support [31]. The conjecture extends Schaefer's theorem of 1978 to every finite domain.

In 2008 Maróti and McKenzie proved that a locally finite idempotent variety is Taylor if and only if it has a weak near-unanimity term [24, 31]. Combined with the hardness theorem of Bulatov, Jeavons and Krokhin, this shows that a problem CSP(Γ) with no weak near-unanimity polymorphism is NP-complete [31]; Zhuk states it as Maróti and McKenzie's [24]. The hypothesis is Taylor's terms (Taylor 1977). According to Barto and Kozik, the proof uses the tame congruence theory of Hobby and McKenzie (1988) [31].

In 2017 Bulatov and Zhuk gave independent proofs of the dichotomy [30, 35, 34]. Zhuk's algorithm reduces domains through binary absorbing subuniverses, centres, polynomially complete quotients and linear equations over Z_p, as described in his own paper [24]. It appeared in the Journal of the ACM 67(5) in 2020, according to Wikipedia and CORDIS [30, 37]. Bulatov's proof appeared at FOCS 2017, pp. 319–330, and as arXiv 1703.03021 [34]. His own list marks it as a best paper and gives the journal version as submitted, as of 29 October 2026 [36]. The proofs establish the Feder–Vardi conjecture. Zhuk's own text says his proof uses the weak near-unanimity terms of Maróti and McKenzie and Rosenberg's classification [24].

In 2024 Zhuk gave a simplified proof: a finite constraint satisfaction problem is tractable if and only if it has infinitely many polymorphisms that are symmetric on all two-element sets [38]. The proof uses strong subalgebras and linear congruences defined globally. In the paper's words, it gives "a new proof of the correctness of Zhuk's algorithm for all tractable CSPs", so it builds on his work of 2017. It is a preprint (version 1 of 1 April 2024, version 2 of 21 October 2024) with no journal reference as of 29 October 2026 (note 6).

Notes

1. Only Wikipedia states that Whitehead's *Treatise* has "no results of a general nature". Whitehead's sense of "universal algebra" is modern according to Wikipedia, and different according to MacTutor.
2. Rosenberg's date is 1970. Zhuk [24], Goldstern and Pinsker [11], Pinsker [25] and arXiv 2406.15184 [26] (announced 1965, 90-page proof 1970) all cite Rosenberg 1970 for the six relation types on finite sets. Wikipedia's papers of 1974 and 1976 are later work on infinite sets. Burris and Sankappanavar cite Murskiĭ 1971 for the monoid technique, so an entry "Murskiĭ 1965" could not be confirmed and is left out [3].
3. Mal'cev's date is 1954, Matematicheskii Sbornik 35(77) [14, 15, 16]. Freese places Pixley's and Jónsson's distributivity terms in the "early 1960's" [16]; Jónsson's paper is Mathematica Scandinavica 21 (1967) [23].
4. The pages 49–104 sometimes given for McKenzie's main paper on Tarski's problem could not be confirmed.
5. Open questions in this chain: Schaefer's own reliance on Post's lattice (later re-proofs use it). Tame congruence theory (1988), bounded width (2009/2014), Jeavons' connection between polymorphisms and invariants (1998) and Bulatov's three-element result (2006) are not entered as links. Lawvere's categorical universal algebra (1963) had only one source and is not entered either.
6. Zhuk's paper of 2024 rests on the preprint alone; no independent account or later result was found.

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Linear algebra

Linear algebra studies systems of linear equations, the arrays of coefficients that describe them, and the spaces on which they act. Its chain starts with elimination on a counting board in the Chinese *Nine Chapters*, a method Europe later reinvented (Newton, Gauss) without any known transmission. Determinants came first, found in parallel by Seki and Leibniz and then developed by Cramer, Laplace and Cauchy. Matrices became objects in their own right only in 1850–1858, followed by canonical forms, the singular value decomposition (SVD) and Perron–Frobenius theory.

In the twentieth century the chain splits into three lines. One concerns eigenvalue inequalities, from Weyl to Knutson and Tao. A second concerns permanents, where van der Waerden's conjecture was proved in 1979–81. A third concerns algorithms: the SVD, and Strassen's race for the exponent ω of matrix multiplication. Interlacing of eigenvalues also became a proof engine, for the Kadison–Singer problem (2013) and the sensitivity conjecture (2019). As of 28 October 2026 the record is $\omega < 2.371177$, from a preprint of 17 August 2026 that used AlphaEvolve. In all, the chain has 31 links.

Elimination and the first determinants

The *Nine Chapters on the Mathematical Art* dates from about 200–100 BCE, with a commentary by Liu Hui in the third century CE (note 1) [1, 2, 3]. It solves linear systems by elimination on a counting board, the method called fangcheng. Columns are multiplied and subtracted until the system is triangular, and the unknowns are then found by back-substitution. No proof is given. Problem 1 of chapter 8 asks for $3x + 2y + z = 39$, $2x + 3y + z = 34$, $x + 2y + 3z = 26$ (a quick check: the solution is $x = 37/4$, $y = 17/4$, $z = 11/4$).

In the *Ars Magna* of 1545 Cardano gave the *regula de modo* for systems of two equations [1, 4]. It expresses each unknown as a ratio of cross-products, which is Cramer's rule without the last step. It rests on no earlier link in this chain.

Newton taught successive elimination of unknowns as a general lesson in his notes of 1669–70, published in 1707 [1, 2, 3]. There is no arrow from the *Nine Chapters*: elimination "was invented independently in several cultures" [2]. The European line is a rediscovery, not a transmission.

Determinants appeared twice, independently. In 1683 Seki Takakazu used determinants of arrays from 2×2 to 5×5 [1, 4]. According to MacTutor he used them to solve equations, not systems of linear equations [1]. In 1693, in a letter to l'Hôpital, Leibniz stated that the vanishing of the coefficient determinant is the condition for three equations to be solvable, and wrote it in index notation [1, 4]. The two ideas "originated separately" [4]; neither rests on the other.

Cramer’s rule and the theory of determinants

Maclaurin proved Cramer’s rule for systems of two and three equations and indicated the case of four, in his *Treatise of Algebra* [1, 5]. The book appeared posthumously in 1748, but a manuscript of Part I is dated 1729 and is almost identical to it [6]. His rule generalises Cardano’s cross-products of 1545. No source states that Maclaurin used Cardano; the link is inferred from MacTutor’s account (note 2).

In 1750 Cramer stated the general rule for n equations in n unknowns, without proof, in an appendix [1, 5]. On the *Nine Chapters* system the rule works as expected (a quick check: $\det A = 12$, and $x_i = \det(A_i)/\det(A)$ gives 37/4, 17/4, 11/4, as elimination does). Cramer’s rule extends Maclaurin’s special cases to n unknowns. This dependency is inferred from the dates; no source says Cramer used Maclaurin’s work [6].

Bézout (1764), Vandermonde (1771) and Laplace (1772) gave methods for computing determinants [1, 4]. Laplace expanded a determinant in complementary minors. The dates of Bézout and Vandermonde vary between sources (note 3). Here there is no arrow to Cramer: Laplace called Cramer’s and Bézout’s methods impractical, and no source says these methods were built on Cramer’s rule [1]. (A quick check: the Leibniz sum over permutations and the Laplace expansion agree on a random integer 4×4 matrix, numpy default_rng(1) with entries in [−5, 5]; both give −588.)

Gauss named the “determinant” in 1801, but for the discriminant of a quadratic form, not in today’s sense [1, 4]. In 1810, working on the orbit of the asteroid Pallas, he devised a notation for symmetric elimination of the least-squares normal equations [1, 2, 3]. This was again a rediscovery of elimination, with no arrow from the *Nine Chapters*.

In 1812 Cauchy gave the modern determinant and the first proof of the product theorem $\det(AB) = \det A \cdot \det B$ [1, 4]. Binet stated the theorem at the same meeting, with a less satisfactory proof. A check of the Cauchy–Binet formula on a random 2×3 times 3×2 integer pair (numpy default_rng(2), entries in [−5, 5]) gives $\det(PQ) = 243$, equal to the sum of products of 2×2 minors. According to MacTutor, Cauchy “reproved earlier results”, so his work builds on Bézout, Vandermonde and Laplace [1].

Cauchy returned to the subject in 1826 and 1829. In 1826 he showed that every real symmetric matrix can be diagonalised, calling the coefficient array a “tableau” [1]. In 1829 he generalised Lagrange’s approach for 3×3 arrays to n×n [7] and noted that the roots of the secular equation are real [8] (note 4). This rests on Lagrange’s 3×3 diagonalisation, and on Cauchy’s own 1812 determinants through the characteristic equation, as MacTutor describes [1]. (A quick check: the largest imaginary part among the eigenvalues of a random symmetric 6×6 matrix, numpy default_rng(3), $G + G^{T}$ with G standard normal, is 0.0.)

In 1841 Jacobi wrote three treatises on determinants, which gave the first algorithmic definition [1]. In the same year he introduced the functional determinant that now bears his name [4]. No earlier link is named as its basis.

Matrices as objects and their canonical forms

In 1850 Sylvester coined the word “matrix”, for “an oblong arrangement of terms” [1, 9]. This is terminology, not a theorem.

Four links from the Algebra chain belong here as well [1, 10, 11]. Grassmann's *Ausdehnungslehre* (1844) treated extension in space. Cayley's matrix algebra followed in 1858 (note 5); he proved the Cayley–Hamilton theorem for 2×2 matrices and checked it for 3×3. Frobenius proved the general Cayley–Hamilton theorem and introduced rank in 1878, and Peano gave the axioms of a vector space in 1888. A symbolic check confirms Cayley–Hamilton for a generic 3×3 matrix: $p(A) = 0$. There is no arrow from Sylvester's word to these proofs, which do not use it as a step.

In 1870, in the *Traité des substitutions*, Jordan gave the canonical form now named after him [1, 7]. He worked with linear substitutions over a finite field of prime order, and extended the form to complex matrices in 1871 [7] (note 6). The modern proof splits the space into generalised eigenspaces and builds Jordan chains in each [12]. A quick check: for $M = P \cdot \mathrm{diag}(1, 2, J_2(4)) \cdot P^{-1}$ with $P = [[1,1,0,0],[0,1,1,0],[0,0,1,1],[1,0,0,2]]$, sympy returns the form $\mathrm{diag}(1, 2, J_2(4))$. Weierstrass's elementary divisors of 1868 may lie behind this, but only Brechenmacher says so, so no arrow is drawn (note 7).

Singular values were found several times independently: by Beltrami (1873), Jordan (1874), Sylvester (1889), Schmidt (1907), Autonne (1915) and Weyl [13, 14]. They are the complete invariants of a bilinear form under orthogonal changes of coordinates, found by diagonalising $A^{T}A$ and AA^{T} together. Schmidt worked with integral operators. These were parallel discoveries: Stewart calls Jordan's work independent of Beltrami's, and says Sylvester worked "in ignorance of his predecessors" [14] (note 8).

In 1907 Perron proved that a positive matrix has a simple positive dominant eigenvalue, with an eigenvector whose entries are all positive [15, 16]. It appeared in Mathematische Annalen 64. The sources consulted do not describe Perron's own method. (A quick check: for a random positive 5×5 matrix, numpy default_rng(4) with entries uniform in [0.01, 1], the spectral radius 3.101627 is an eigenvalue with an all-positive eigenvector.)

In 1912 Frobenius extended the theorem to irreducible non-negative matrices [15, 16]. The eigenvalues on the spectral circle are then the radius times roots of unity, whose number is the index of imprimitivity. Both sources call this an extension of Perron's theorem, so it builds on Perron 1907, though neither says the proof uses Perron's. According to MacTutor, Frobenius's own work on rank of 1878 was an influence [1].

Eigenvalue inequalities and permanents

Also in 1912 Weyl proved inequalities for the eigenvalues of a sum $A + B$ of Hermitian matrices, "generally proven by minimax methods" [17, 18]. He also asked which spectra $A + B$ can have. The inequalities read $\lambda_{i+j-1}(A + B) \le \lambda_i(A) + \lambda_j(B)$ (a quick check: they hold for all $i + j \le 5$ on a random Hermitian 4×4 pair, numpy default_rng(5), $(G + G^{*})/2$ with complex normal G; the trace identity holds up to rounding error of order 10^{-15}). Weyl uses the fact that these eigenvalues are real, going back to Cauchy's work of 1826; this is inferred from the content, not stated by a source (note 9).

In 1926 van der Waerden posed "Aufgabe 45" in the Jahresbericht der DMV (35, p. 117, according to van Lint) [19, 20, 21]. He conjectured that $\mathrm{per}(A) \ge n!/n^{n}$ for every doubly stochastic n×n matrix A, with equality only for the matrix J/n whose entries all equal 1/n. A quick check: $\mathrm{per}(J_5/5) = 0.0384 = 5!/5^{5}$, and a random 5×5 matrix

(numpy default_rng(6), entries uniform in [0.01, 1]) balanced by 1000 Sinkhorn steps has permanent 0.044527 ≥ 0.0384.

In 1962 Horn conjectured that the possible spectra of A + B are cut out by the trace identity and finitely many linear inequalities, defined recursively [17, 22]. The paper appeared in Pacific J. Math. 12, 225–241. It answers Weyl's question of 1912. How far Horn computed the inequalities differs between sources (note 10).

Egorychev and Falikman proved van der Waerden's conjecture independently, around 1979, with papers in 1980–81 [19, 20]. Egorychev's proof applies the Alexandrov–Fenchel inequality [20]. The two shared the Fulkerson Prize in 1982 [20]. The conjecture had waited 1981 − 1926 = 55 years (note 11).

Klyachko (1998, Selecta Math. 4, 419–445) found a recursive relation between the classical problem on A + B and the quantum problem for tensor products [17, 18]. As a result, the saturation conjecture would imply Horn's conjecture. His argument uses stability results of Totaro and of Kempf and Ness [18]. It rests on Horn's work of 1962.

In 1999 Knutson and Tao proved the saturation conjecture with combinatorial objects they called honeycombs, so Horn's conjecture follows [17, 22]. The paper appeared in J. AMS 12, 1055–1090. It rests on Klyachko's reduction and settles Horn's conjecture, 1999 − 1962 = 37 years after it was posed; counting to Klyachko gives 1998 − 1962 = 36 (note 12).

The SVD and fast multiplication

In 1936 Eckart and Young gave the first proof of the SVD for rectangular and complex matrices, as a generalised principal-axis transformation [13]. They also proved that truncating the SVD gives the best low-rank approximation. Stewart shows this rediscovers Schmidt's approximation theorem of 1907, so there is no arrow from the earlier SVD work of 1873–1915 [14] (note 8). A quick check: for a random 6×5 matrix (numpy default_rng(7), standard normal), in the Frobenius norm $\|X - X_1\| = 3.8089663447 = \sqrt{(\Sigma\ \sigma_i^2 \text{ for } i \geq 2)}$.

Golub and Kahan (1965) and Golub and Reinsch (1970) gave a numerically stable algorithm for the SVD [13, 14]. It reduces the matrix to bidiagonal form by Householder reflections, replacing Jacobi-style sweeps with orthogonal reductions. Golub–Reinsch is still the most used method; Stewart calls it "the workhorse" [14]. A quick check of orthogonality: for Q from the QR factorisation of a random 50×50 matrix (numpy default_rng(8)), $\|Q^TQ - I\| = 4.4 \times 10^{-15}$. The algorithm brings the SVD into numerical analysis; a direct link to Eckart and Young is not stated by a source (note 13).

In 1969 Strassen multiplied 2×2 block matrices with 7 products instead of 8 [23, 24]. Applied recursively, this gives $\omega \leq \log_2 7 \approx 2.8074$. A quick check: the 7-product formula equals the ordinary product on a random 2×2 pair (numpy default_rng(9)), and $\log_2 7 = 2.807354922$. It rests on no earlier link.

From 1978 on, each new bound on ω built on Strassen's and on the bounds before it [23, 24, 25, 26]. The tools were border rank, the asymptotic sum inequality and the laser method. The bounds, following Wikipedia's list, were 2.796 (1978), 2.780 (1979), 2.522, 2.517 and 2.496 (all 1981), 2.479 (1986), 2.3755 (1990), 2.3737 (2010), 2.3729 (2012), 2.3728639 (2014), 2.3728596 (2020), 2.371866 (2022), and 2.371552

and 2.371339 (2024) [23]. Le Gall gives rounded values and some different years (note 14). The 2024 bound improves Strassen's $\log_2 7$ by 0.436016.

On 17 August 2026 Dupont, Eisenberger, Kozlovskii, Mehrabian, Ruiz, See, Zhou, Alman, Vassilevska Williams and Balog posted a preprint proving $\omega < 2.371177$ [33]. They used the laser method with an analysis of combination loss, reformulated in a larger setting. The resulting problem was solved with an optimiser designed by machine learning and refined with AlphaEvolve. As of 28 October 2026 it is a preprint (arXiv v1). It improves the 2024 bound 2.371339 within the laser-method framework (a quick check: $2.371339 - 2.371177 = 0.000162$) (note 15).

Interlacing as a proof engine

In 2013 Marcus, Spielman and Srivastava solved the Kadison–Singer problem of 1959 [27, 28, 29]. They proved Weaver's conjecture KS_2, which gives Anderson's paving conjecture. The method uses interlacing families of polynomials to bound the largest root of a mixed characteristic polynomial. The work won the Pólya Prize in 2014 and was published in Annals of Mathematics 182 (2015), 327–350. The problem had been open $2013 - 1959 = 54$ years.

In 2019 Huang proved that every induced subgraph of the n-cube Q_n on $2^{n-1} + 1$ vertices has maximum degree at least $\sqrt{n}$ [30, 31, 32]. By the work of Gotsman and Linial this proves the sensitivity conjecture of Nisan and Szegedy (1992). The proof uses a signed adjacency matrix A_n with $A_n^2 = nI$, together with Cauchy's interlacing theorem. A quick check: $A_n^2 = nI$ for $n = 3, 4, 5$, with largest eigenvalues 1.732051, 2.0 and 2.236068, equal to $\sqrt{n}$. It was published in Annals of Mathematics 190 (2019), 949–955.

Notes

1. The dating of the *Nine Chapters* is uncertain. MacTutor gives 200–100 BCE [1]. Wikipedia says the first reference to the title dates from 179 CE, with parts perhaps from about 150 BCE [2]. Grcar dates the problem only to "before 200 AD" [3].
2. Only MacTutor's account supports Maclaurin's reliance on Cardano. Kosinski calls Maclaurin's rule for four equations nonsensical: it gives no correct sign rule and ignores a vanishing denominator [6].
3. MacTutor and Wikipedia both date Vandermonde to 1771; the often-quoted 1772 is not sourced. MacTutor dates Bézout's determinant methods to 1764, Wikipedia to 1779 [1, 4].
4. Cauchy's contribution is two papers: the diagonalisation of 1826 described by MacTutor, and the 1829 paper that generalised Lagrange's 3×3 approach and noted the real roots of the secular equation [7, 8].
5. Cayley's matrix algebra is dated 1858 in MacTutor's article on matrices and 1857 in its article on linear spaces [1, 10].
6. Jordan's form of 1870 is over a prime field, not the complex form of today [1, 7].
7. Weierstrass's elementary divisors of 1868 have one source, Brechenmacher, and the dispute between Jordan and Kronecker is described only in Brechenmacher's papers [8].
8. Stewart credits Autonne with complex matrices in 1913 (Wikipedia: 1915), and says the "Eckart–Young" approximation theorem is Schmidt's of 1907 [13, 14].

9. Weyl’s reliance on Cauchy is an inference, as is Frobenius’s use of Perron’s proof.
10. For the range of Horn’s computations, AMS Notices says $n \le 8$, while Fulton calls $n \le 5$ “the region that Horn investigated most thoroughly” [17, 22].
11. The Encyclopedia of Mathematics says the conjecture was solved “around 1979”, with papers in 1980–81. Wikipedia dates both proofs to 1981 and also lists a proof by Gyires in 1980, which was not checked [19, 20].
12. Knutson and Tao say Horn’s conjecture “waited thirty-six years”: $1998 - 1962 = 36$ counts to Klyachko, $1999 - 1962 = 37$ to saturation [17].
13. No source states a link from Golub and Kahan to Eckart and Young.
14. Le Gall gives rounded values and some different years: Pan 1979, and Coppersmith–Winograd 1982 and 1987, using conference dates, where Wikipedia has 1978, 1981 and 1990. He lists Schönhage at < 2.55 where Wikipedia has 2.522. This chapter follows Wikipedia [23, 24]. Wikipedia dates Alman–Vassilevska Williams to 2020 and Duan–Wu–Zhou to 2022 (arXiv 2210.10173, October 2022); arXiv 2404.16349 cites Duan–Wu–Zhou as FOCS 2023 and Alman–Vassilevska Williams as [AV21] [25, 26].
15. Only the preprint’s own abstract states how it improves the 2024 bound [33]. AlphaTensor (2022) has a single source, and AlphaEvolve’s 2025 result of 48 multiplications for 4×4 matrices is unconfirmed; neither is a link here.
16. Gershgorin’s theorem (1931), Hadamard’s inequality (1893) and Sylvester’s law of inertia (1852) were computed but not sourced, so they are not links in this chain.

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Associative rings and algebras

This field studies rings and algebras in which multiplication is associative but need not be commutative. The chain starts with Hamilton's quaternions (1843) and Frobenius's proof (1878) that the real numbers R, the complex numbers C and the quaternions H are the only finite-dimensional real associative division algebras. Wedderburn (1905, 1907) and Artin (1927/28) then turned the old "hypercomplex numbers" into a structure theory.

Three lines branch off from there. One is structure: Jacobson (1945), Morita (1958) and Goldie (1958 and 1960). A second is division algebras: the Brauer group and the Albert–Brauer–Hasse–Noether theorem of 1932. A third is a set of problems: Köthe's conjecture (1930), the Amitsur–Levitzki theorem (1950) and the Golod–Shafarevich theorem (1964). In 2026 the newest links are counterexamples. Gardam refuted the unit conjecture in characteristic 2 (Annals, 2021), and two unrefereed preprints of September 2026 claim counterexamples to Köthe's conjecture, so that conjecture is not yet settled. The zero-divisor conjecture is still open. There is no arrow from Wedderburn's little theorem to his structure theorem of 1907, and none from Wedderburn or Artin to Morita. The chain has 21 links in all.

From quaternions to hypercomplex structure

Hamilton found the quaternions on 16 October 1843, with the rule $i^2 = j^2 = k^2 = ijk = -1$ [1, 2]. MacTutor calls them "the first noncommutative algebra to be studied" [1]. They are the only non-commutative finite-dimensional real associative division algebra [3]. A quick check: for 100 random integer triples (random.Random(11), entries in $[-9, 9]$), $(q_1q_2)q_3 = q_1(q_2q_3)$, and the norm is multiplicative, $N(q_1q_2) = N(q_1)N(q_2)$, which is Euler's four-square identity. Multiplication is not commutative: $(1+2i+3j+4k)(5+6i+7j+8k) = -60+12i+30j+24k$, while the reverse order gives $-60+20i+14j+32k$.

In 1878 Frobenius proved that R, C and H are the only finite-dimensional associative real division algebras [3, 4]. The paper appeared in Crelle's Journal 84 (note 1). In the standard proof every element satisfies a real quadratic equation, and the anticommuting imaginary parts force the dimension to be 1, 2 or 4 [4]. Frobenius's own argument was not seen. The theorem includes Hamilton's H as the only non-commutative case; this is a classification, not a use of Hamilton's work as a proof step.

In 1905 Wedderburn proved that every finite division ring is commutative [5, 6]. The proof uses the interplay of the additive and multiplicative groups [5]. The first of his three proofs had a gap [5, 6]. A quick check shows why quaternions give nothing new over a finite field: over F_p the equation $x^2 + y^2 + 1 \equiv 0 \pmod p$ is solvable for all 45 odd primes p below 200, so the quaternions over F_p always have zero divisors; over F_7, for instance, $(i+2j+3k)^2 = -14 = 0$.

Also in 1905 Dickson gave the first correct proof of the little theorem [5, 6]. Wedderburn's second and third proofs came after he had seen Dickson's, according

to Parshall’s analysis of 1983. Dickson’s proof was independent of Wedderburn’s, so there is no arrow between them (note 2).

In 1907 Wedderburn published “On hypercomplex numbers” in the Proceedings of the London Mathematical Society [5, 7]. He proved that every semisimple algebra is a direct sum of simple algebras, each a matrix algebra over a division algebra. The sources consulted do not describe his method. A quick check on group algebras: $\mathbb{C}[S_3] \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$, since $1 + 1 + 4 = 6$, and for S_4 the block sizes give $1 + 1 + 4 + 9 + 9 = 24$. The theorem belongs to the same programme as the little theorem of 1905, but no dependency between them is shown, so there is no arrow.

Artin gave the theorem its final form in “Zur Theorie der hyperkomplexen Zahlen”, Abh. Math. Sem. Hamburg 5, pp. 251–260 [7, 8, 9, 10] (note 3). A ring with the minimum condition on right ideals and no nilpotent ideals is a finite direct sum of full matrix rings over division rings. Both the Encyclopedia of Mathematics and Wikipedia call this a generalisation of Wedderburn’s theorem of 1907, so it builds on Wedderburn. Neither says that Artin’s proof uses Wedderburn’s.

Division algebras and the Brauer group

In the late 1920s Brauer introduced the group now named after him [11, 12, 13, 14]. Its elements are classes of central simple algebras, multiplied by the tensor product, with the inverse of [A] given by the opposite algebra [A^op]. For a finite field Br(F_q) = 0, and for the reals Br(R) = Z/2. MacTutor places this work in his years at Königsberg, 1925–33 [12] (note 4). A quick check: $H \otimes H^{op} \cong M_4(\mathbb{R})$, because the 16 maps $q \mapsto e_i q e_j$ span End_R(H) with rank 16, so [H] has order 2. The construction uses Wedderburn’s theorem of 1907, by which every central simple algebra is $M_n(D)$ for a division algebra D. Wedderburn’s little theorem and Frobenius’s theorem supply the two cases Br(F_q) = 0 and Br(R) = Z/2.

In 1932 Brauer, Hasse and Noether, and Albert, proved that a central simple algebra over a number field which splits at every completion is a matrix algebra [15, 16]. With the Grunwald–Wang theorem, every such algebra is cyclic [15]. According to MacTutor the joint work of 1932 led to “the complete determination of … the Brauer group of an algebraic number field” [16]. The sources do not state the proof method. The theorem rests on the Brauer group (note 5).

Radicals, identities and quotient rings

In 1930 Köthe asked a question that is still central [17, 18]. If a ring R has no non-zero nil ideal, does it also have no non-zero nil one-sided ideal? In 1972 Krempa showed this equivalent to the statement that the polynomial ring R[x] is Jacobson radical for every nil ring R [19]. Wikipedia also gives a form in terms of 2×2 matrix rings $M_2(J)$, citing Krempa but without attributing that form to him [17]. No earlier link is named as its basis.

In his thesis of 1940 Higman posed the unit conjecture, later popularised by Kaplansky [20, 21] (note 6). It states that the group ring K[G] of a torsion-free group G has only trivial units, those of the form kg with k a non-zero scalar and g in G.

In 1945 Jacobson defined the radical of an arbitrary ring as the intersection of its maximal right ideals [22, 23, 24]. He described it through simple modules or through

quasi-regularity, with no chain condition. A quick check: J(Z/12) = {0, 6}, which is also the nilradical. For Artinian rings the Jacobson radical equals the nilpotent radical of Artin's theory, so Artin's work of 1927 was an influence; no source states that Jacobson's proofs use it (note 7).

In 1950 Amitsur and Levitzki proved that the standard polynomial s_{2n} vanishes on the matrix ring $M_n(R)$ over any commutative ring R, and that 2n is the minimal degree [25, 26, 27]. The original proof substitutes matrix units; the Encyclopedia of Mathematics calls it combinatorial, and Procesi "a direct verification" [27, 28]. A quick check on random integer matrices (numpy default_rng(12), entries in $[-3, 3]$): $s_4 = 0$ and $s_3 \neq 0$ for 2×2 matrices, and $s_6 = 0$ and $s_5 \neq 0$ for 3×3. The theorem rests on no earlier link. A later proof by Razmyslov (1974) derives it from the multilinear Cayley–Hamilton theorem [25, 27], which Procesi credits to Formanek [28].

In 1958 Morita showed that the module categories Mod-R and Mod-S are equivalent exactly when $S \cong \mathrm{End}(P)$ for a finitely generated projective generator P [29, 30]. In particular R and the matrix ring $M_n(R)$ are Morita equivalent. Wikipedia adds that the equivalences come from bimodules [29]. There is no arrow from Wedderburn or Artin: Wikipedia relates the example $M_n(R)$ to the Artin–Wedderburn classification, but does not say Morita's proof uses it.

Goldie proved in 1958 (the prime case) and 1960 (the semiprime case) that semiprime right Goldie rings are exactly the rings with a semisimple Artinian classical right ring of quotients [31, 32, 33]. Such a ring is right Ore, so the quotient ring exists [31]. The structure of that quotient ring, a product of simple Artinian rings, uses Artin's theorem; Wikipedia states this as a consequence rather than a proof step [31].

Golod–Shafarevich and the unit conjecture

In 1964 Golod and Shafarevich proved that graded algebras with relatively few relations are infinite-dimensional [34, 35, 36]. This gave infinite class field towers and a counterexample to the general Burnside problem of 1902, which it answers negatively. In the 1964 form, the threshold for r quadratic relations among d generators is $(d - 1)^2/4$: below it the algebra is infinite-dimensional. According to Ershov's survey the sharper threshold $d^2/4$ is a later improvement by Vinberg and Roquette [37] (note 8). A quick check of the series behind the criterion, with d = 4: $1/(1 - 4t + 4t^2)$ ($r = 4 = d^2/4$) has no negative coefficient up to t^{29}, while $1/(1 - 4t + 5t^2)$ (r = 5) has its first negative coefficient at index 6.

In 2021 Gardam refuted the unit conjecture [20, 21]. He found a non-trivial unit in $F_2[P]$, where P is the Hantzsche–Wendt group, by a computer search using Boolean satisfiability. The paper appeared in Annals of Mathematics 194(3). A recomputation takes P as affine maps of R^3, $a(v) = \mathrm{diag}(1, -1, -1)v + (½, ½, 0)$ and $b(v) = \mathrm{diag}(-1, 1, -1)v + (0, ½, ½)$, which satisfy both defining relations $b^{-1}a^2b = a^{-2}$ and $a^{-1}b^2a = b^{-2}$. The unit $\alpha = p + qa + rb + sab$ and its inverse α' each have support 21, $\alpha'\alpha = \alpha\alpha' = 1$, and α is not trivial.

Murray (2021, preprint) extended Gardam's construction to every prime characteristic [38, 39]. As of 29 October 2026 it has no journal reference. In 2023 Gardam showed in "Non-trivial units of complex group rings" that the conjecture also fails in characteristic 0 [40]. Wikipedia says this uses "essentially the same element", so it builds on the unit of 2021 [41]; the preprint's own abstract states only that the conjecture fails. It too has no journal reference as of 28 October 2026.

On 23 March 2026 Dietrich, Lee, Nies and Vinyals posted “On the trivial units property and the unique product property” [42]. They classify certain symmetric non-trivial units in F_2[Hantzsche–Wendt] and propose a new candidate group that fails the unique units property but may satisfy the trivial units property. The abstract states that the Hantzsche–Wendt group “was used in Gardam’s refutal of Kaplansky’s unit conjecture”, so the work builds on Gardam 2021. A Sydney seminar abstract by Nies (30 January 2026) confirms the search for units on the Hantzsche–Wendt group and a group failing the unique product property, F(4,3), found by SAT solvers [43] (note 9).

Köthe’s conjecture in 2026

On 7 September 2026 Adamczewski, Böhmler and Marczinzik posted “A counterexample to Köthe’s conjecture and a question of Rowen” [44]. They claim a nil algebra whose 2×2 matrix ring is not nil, so that two nil left ideals have a non-nil sum. They use the example to answer a question of Rowen as well. A Lean 4 disproof covers only Krempa’s matrix form, and the repository’s README says that a pre-release AI system found the proof [45]. The claim rests on Köthe’s problem of 1930, through Krempa’s form. It is unrefereed as of 28 October 2026 (note 10).

On 14 September 2026 Greenfeld, King and Vendramin posted “The Köthe conjecture via point modules”, which claims a short refutation [46]. It builds on the point-module approach of Greenfeld, King, Easton and Ford, and its abstract does not mention the earlier preprint. According to the authors’ own preprint the conjecture is refuted; no independent source yet confirms it, and Wikipedia still lists the conjecture as open [17]. It is unrefereed as of 29 October 2026, and whether it depends on the September 7 preprint is open (note 10).

Notes

1. Wikipedia dates Frobenius’s theorem 1877 and the Encyclopedia of Mathematics 1878 (Crelle 84). Most likely it was found in 1877 and printed in 1878 [3, 4].
2. Wedderburn has priority for the little theorem (1905), but Dickson gave the first correct proof, according to Parshall as reported by MacTutor and Wikipedia [5, 6].
3. Artin’s paper is “Hamb. Abh. 5 (1928)” in Brauer’s bibliography of 1967 and “around 1928” in MacTutor [9, 10]. A December 1927 date credited to Springer could not be found again, so this chapter writes 1927/28. Artin–Nesbitt–Thrall is dated 1948 by MacTutor and by Brauer’s bibliography; the Encyclopedia of Mathematics’ 1946 was not seen again.
4. The year in which Brauer introduced the Brauer group is not fixed by the sources; MacTutor places it in his Königsberg years, 1925–33 [12].
5. Wikipedia says Albert proved the Albert–Brauer–Hasse–Noether theorem independently [15]. MacTutor’s page on Hasse does not mention Albert, so an earlier statement that MacTutor gives Brauer, Hasse and Noether priority is not confirmed [16].
6. Gardam attributes the unit conjecture to Kaplansky, and Quanta to Higman’s thesis of 1940. Both fit, since Higman posed it and Kaplansky popularised it [20, 21].
7. Jacobson’s debt to Artin is not stated by a source.

8. Ershov credits the $d^2/4$ improvement to Vinberg and Roquette, not to Gaschütz; an earlier attribution to "Vinberg and Gaschütz 1967" was not confirmed. Only Ershov's survey states this [37].
9. The Sydney seminar abstract is the authors' own and does not mention the new candidate's trivial units [43].
10. Whether the two Köthe preprints are independent is open: neither abstract cites the other. Whether either survives refereeing is also open. Wikipedia still calls Köthe's conjecture "open as of 2025", which is out of date rather than a conflict of fact [17]. The GitHub repository is the first author's own, not an independent source [45].
11. An earlier zero-divisor check over F_7, $(i+2j+3k)(i-2j-3k)$, was wrong; $(i+2j+3k)^2 = -14 = 0$ is right. The zero-divisor conjecture is still open as of January 2025 [39]. The status of the idempotent conjecture rests on Wikipedia ("as of 2021") [41]. The link to the Kurosh problem is not drawn.

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Non-associative algebras

Non-associative algebras are algebras in which (xy)z need not equal x(yz). The field has two nineteenth-century roots. The first is the composition algebras: Degen's eight-square identity, the octonions of Graves and Cayley that followed Hamilton's quaternions, and Hurwitz's theorem that closes the list at dimensions 1, 2, 4 and 8. The second is the Lie algebras: Lie's infinitesimal groups, Killing's classification with less rigorous proofs, Cartan's rigorous rewriting, and Weyl's representation theory.

Jordan algebras (1932–34) tie back to the octonions through the 27-dimensional exceptional algebra, and Malcev algebras (1955) through Moufang loops. Zelmanov's Jordan theory fed his solution of the restricted Burnside problem (1989–91). The latest link is the classification of simple Lie algebras in characteristic $p > 3$, announced complete in 2006; no milestone of 2025–2026 was found. There is no arrow from Lie to Killing, who found the concept independently, nor from Degen to Graves. The chain has 21 links in all.

Composition algebras: from Degen to Hurwitz

In 1818 Degen found the eight-square identity: the product of two sums of eight squares is again a sum of eight squares, given by an explicit polynomial identity [1, 2]. It rests on no earlier link. The identity is equivalent to the statement that the octonion norm is multiplicative (a quick check: with the Cayley–Dickson product, this holds for 1000 out of 1000 random integer pairs, seed 1818, entries in $[-9, 9]$).

Hamilton's quaternions of 1843 are treated in the chapter on associative rings and algebras [3, 4]. They were the starting point for the next step.

In December 1843 John Graves found the octaves, an 8-dimensional normed division algebra [1, 5, 6, 7]. He described them in a letter to Hamilton of 26 December 1843 and built them with the eight-square identity. The octonions are not associative, but they are alternative: $(o_1 o_1) o_2 = o_1 (o_1 o_2)$ (a quick check: for 1000 random integer triples, seed 1843, entries in $[-9, 9]$, all 1000 are non-associative and all satisfy the alternative law). Graves's work rests on Hamilton's quaternions as inspiration. He rediscovered Degen's identity of 1818 rather than using it, so there is no arrow from Degen.

In July 1844 Hamilton wrote to Graves that $A \cdot BC = AB \cdot C$ holds for quaternions, "but not so, generally, with your octaves" [1]. The remark is a comment on Graves's construction of 1843 (note 1).

In 1845 Cayley published the octonions independently in the *Philosophical Magazine*, hence the name "Cayley numbers" [1, 5, 6, 8]. His discovery did not depend on Graves's, which was not yet published (note 2).

In 1898 Hurwitz proved that a positive definite real quadratic form admits composition only in dimensions 1, 2, 4 and 8 [8, 15]. The first paper, on the composition of quadratic forms, appeared in 1898, and the full paper posthumously in 1923. The octonions are the dimension-8 case of the problem, but no source says the proof uses them, so they count as an influence, not an arrow. Beyond dimension 8 composition fails (a quick

check: the 16-dimensional sedenions have a non-multiplicative norm for 100 out of 100 random integer pairs, seed 1898, entries in $[-9, 9]$; and with the Cayley–Dickson convention $(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c})$, the product $(e_3 + e_{10})(e_6 - e_{15}) = 0$ exhibits a zero divisor).

Lie algebras: Lie, Killing and Cartan

From 1873–74 Lie studied continuous transformation groups systematically [9, 10]. He built the "infinitesimal group" from infinitesimal transformations; this was later called a Lie algebra. With Engel he wrote the *Theorie der Transformationsgruppen*, in three volumes (1888–1893). It rests on no earlier link in this chain.

Between 1884 and 1890 Killing introduced Lie algebras on his own and classified the simple ones over the complex numbers [9, 11, 12]. In 1886 he conjectured, wrongly, that only the algebras sl and so are simple. He found the exceptional algebra "G" on 23 May 1887 and the complete list on 18 October 1887, by root decomposition with respect to a maximal Cartan subalgebra. The work was published in Mathematische Annalen in four parts. A quick check from the Cartan matrices: G_2 has 12 roots (dimension 14), F_4 48 (52), E_6 72 (78), E_7 126 (133) and E_8 240 (248), with Cartan determinants 1, 1, 3, 2 and 1. There is no dependency on Lie: when Killing wrote his Programmschrift of 1884 he "was not aware of Lie's work" [11]. His own "G" refuted his conjecture of 1886 (note 3).

Cartan's thesis of 1894 gave a rigorous foundation for the local theory, completed Killing's classification and constructed each exceptional simple Lie algebra [11, 12, 13, 14]. It rests directly on Killing. The thesis is "essentially a rigorous rewriting of Killing's paper" [14]; Cartan "reworked Killing's proofs" [11] and constructed the exceptional algebras whose existence Killing had not proved [13]. A quick check of a central tool: the Killing form of sl_2 in the basis (E, H, F) is [[0,0,4],[0,8,0],[4,0,0]], with determinant $-128 \neq 0$, so it is nondegenerate.

In 1913, in his work on the linear representations of the simple groups, Cartan found spinors [13, 16]. No earlier link is named as its basis.

In 1924–26 Weyl proved complete reducibility of representations through compact real forms, the "unitarian trick", and gave the general character formula [17, 18, 19, 20, 21]. He extended the invariant integration of Hurwitz and Schur [19, 20]. The proof also uses the semisimple algebras of Killing and Cartan, through their compact real forms; no source says it uses their structure theory directly (note 4).

In 1934 Weyl gave the name "Lie algebra" [9, 22]. This is terminology, not a theorem.

Jordan and Malcev algebras

In 1932 Pascual Jordan proposed a programme, and in 1933 he introduced algebras for the observables of quantum mechanics [23, 24, 25]. They use the commutative product $x \circ y = (xy + yx)/2$ and satisfy the Jordan identity $(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x))$. A quick check on random symmetric 4×4 matrices (seed 1933, 100 triples $M + M^{T}$ with standard normal entries): the Jordan identity holds and $\circ$ is not associative. No earlier link is named as its basis.

In 1934 Jordan, von Neumann and Wigner classified the finite-dimensional formally real Jordan algebras [1, 23, 25]. There are four infinite families and one exception,

the 3×3 hermitian octonionic matrices, which Albert showed is truly exceptional. Its dimension is 27: three real diagonal entries plus three octonion entries of dimension 8, 3 + 3·8 = 27. The classification rests on the problem Jordan posed in 1933, and uses the octonions as building material.

Albert studied Jordan algebras systematically in papers of 1946, 1947 and 1950, and first used the name “Jordan algebra” in 1946 [23, 25, 26]. Wikipedia credits him with the general Peirce decomposition in 1947 [23] (note 5). No earlier link is named as its basis.

In 1955 Mal’cev introduced the algebras now named after him, the tangent algebras of smooth (analytic) Moufang loops [27, 28, 29]. The imaginary octonions under the commutator [x, y] = xy − yx are the prototype. A quick check on 200 random integer triples (seed 1955, entries in [−5, 5]): for the imaginary octonions the Jacobi identity fails, while the Malcev identity J(x, y, [x, z]) = [J(x, y, z), x] holds. The theory rests on analytic Moufang loops [27, 29]. The octonions serve only as the prototype example; the commutator algebra of any alternative algebra is a Malcev algebra, but no source says the proofs of 1955 use the octonions.

Burnside, Zelmanov and characteristic p

Kostrikin worked on the restricted Burnside problem for prime exponent, which Wikipedia dates to 1958 [30, 31, 32]. MacTutor and Aznar date his proof of the local nilpotency of Engel Lie algebras to 1959 (note 6). The method uses identities in Lie algebras of finite characteristic. The proof was not fully satisfactory, and a corrected version came much later [31, 32]. It partly answers Burnside’s problem of 1902. According to MacTutor it rests on Magnus’s reduction of the prime case to Engel Lie algebras [31].

McCrimmon introduced quadratic Jordan algebras, an axiomatisation built on the quadratic operator U_x instead of the bilinear product [23, 33]. According to Jacobson this gave, for the first time, a satisfactory structure theory in characteristic 2 [33]. Wikipedia dates it to 1966 [23] (note 7). No earlier link is named as its basis.

In 1979 Zelmanov showed that even in infinite dimensions there are no simple exceptional Jordan algebras other than Albert algebras [23, 25, 31, 32]. In 1983 he showed that every simple Jordan algebra is hermitian, of spin type or an Albert algebra, and presented the work at the ICM in Warsaw (note 8). This extends the classification of Jordan, von Neumann and Wigner and the finite-dimensional theory to infinite dimensions [31, 32]. It may also use McCrimmon’s quadratic Jordan algebras, but no source says so.

In 1987 Zelmanov proved that the Engel identity implies nilpotency for Lie algebras [30, 31, 32]. In 1989 he reduced the restricted Burnside problem to Engel Lie algebras, and in 1991 he solved it for every exponent, with papers in 1990 and 1991 (note 9). The proof uses a deep structure theory of quadratic Jordan algebras and the local nilpotency of sandwich algebras [31, 32]. According to MacTutor it also uses divided powers [31]. He received the Fields Medal in 1994. The solution rests on Burnside’s problem and Kostrikin’s prime case, and on the quadratic Jordan theory of McCrimmon and of Zelmanov himself. Burnside posed his problem 1991 − 1902 = 89 years earlier.

The last link is the classification of finite-dimensional simple Lie algebras over algebraically closed fields of characteristic p > 3 [34, 35, 36, 37]. Each is classical,

a filtered Lie algebra of Cartan type, or a Melikian algebra (which occurs only for p = 5). The method is local analysis of the root structure relative to tori of maximal dimension. Block and Wilson did the restricted case for p > 7 (1988), Strade the general case for p > 7, and Premet and Strade the remaining cases p = 5 and 7. The completion was announced in arXiv math/0601380 in January 2006, and the last step appeared in J. Algebra 320 (2008), 3559–3604 (note 10). The “classical” type comes from the characteristic-0 theory of Killing and Cartan; no source says the proof uses that theory. The classification theorem covers only p > 3: for characteristics 3 and 2, Premet and Strade describe further algebras (Skryabin’s and Brown’s) and exceptions to the isomorphism theorem, without a complete list [34].

Notes

1. Hamilton’s remark of 1844 rests on Baez alone; Wikipedia, Oregon State, Elduque and MacTutor do not mention it [1].
2. Graves found the octonions in December 1843 but published after Cayley (1845), so the name “Cayley numbers” reflects publication, not discovery. Wikipedia, Oregon State and Elduque credit both as independent discoverers [5, 6, 8].
3. MacTutor says Killing “was not aware of Lie’s work” when he wrote his Programmschrift of 1884, and Wikipedia and the Columbia notes agree. His correspondence with Engel began in November 1885 [9, 11, 12].
4. Weyl’s use of the Killing–Cartan structure theory is an inference from the content.
5. Only Wikipedia states that Albert gave the general Peirce decomposition in 1947 [23].
6. Wikipedia says Kostrikin proved the prime-exponent case in 1958; MacTutor and Aznar say he proved the local nilpotency of Engel Lie algebras in 1959. All agree the proof needed a later correction. These may be two results [30, 31, 32].
7. Only Wikipedia gives the year 1966 for McCrimmon’s quadratic Jordan algebras [23].
8. Zelmanov’s Jordan work is two results: the McCrimmon minicourse gives 1979 (no simple exceptional algebras beyond Albert algebras) and 1983 (the classification of simple Jordan algebras), which MacTutor and Aznar tie to his ICM lecture of 1983 in Warsaw [25, 31, 32].
9. MacTutor and Aznar give the reduction in 1989 and the solution in 1991; Wikipedia says 1989 and lists papers of 1990 and 1991 [30, 31, 32].
10. The classification was announced complete in January 2006 (arXiv math/0601380), with the last step in J. Algebra 320 (2008); Wikipedia’s “2004” is looser [34, 35, 36].
11. Several topics are not entered as links: Kac–Moody algebras, Dynkin diagrams, Serre’s relations, Chevalley bases, Engel’s and Ado’s theorems, and Lie superalgebras. Bott–Milnor and Kervaire (1958) and Adams (1960) are not sourced. Euler’s four-square identity (1748) was computed but not sourced. Cartan’s real forms are undated in the sources consulted.

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Algebraic geometry

Algebraic geometry studies the solution sets of polynomial equations: curves, surfaces and their higher-dimensional analogues. Its chain starts with Bézout's count of intersection points (1779) and the Riemann–Roch theorem for curves (1857–1865). It runs through the Italian classification of surfaces (1891–1914) and is rebuilt on commutative algebra by Zariski and Weil in the 1930s and 1940s.

From there two main lines run. One is resolution of singularities: Zariski (1939 and 1944), Hironaka (1964), and the case of characteristic p in dimension 4 and above, which is still open. The other is birational classification: Mori (1979–1988), Birkar, Cascini, Hacon and McKernan (2010), and Birkar (2016). In 2026 the newest link is the preprint on Hodge atoms (2025), which shows that a very general cubic fourfold is not rational. There is no arrow from Hironaka's proof to the resolutions in characteristic p, and the Italian proofs of resolution carry no arrow because Zariski showed them incomplete. The chain has 35 links in all.

From Bézout to Riemann–Roch

In 1779 Bézout showed that the number of common solutions of n complete equations is the product of their degrees [1, 2, 3, 4, 5]. This was the first satisfactory proof of a statement by Maclaurin. The proof combines elimination with an induction that involves "a little hand waving" [1]. A quick check: the conics $x^2 + y^2 = 1$ and $x^2/4 + 4y^2 = 1$ meet in 4 points $(\pm 2/\sqrt{5}, \pm 1/\sqrt{5})$, the product of the degrees; the tangent line $y = 1$ meets the circle where $x^2 = 0$, in one point of multiplicity 2. The work rests on Maclaurin's statement about intersections; a Leiden thesis says it came "as a result of earlier work of Colin Maclaurin" [1, 5]. The gap in the proof is confirmed by several sources [1, 3, 4]. The notions of intersection multiplicity and conservation of number were put on firm ground by van der Waerden in 1927–28, after which the general theorem is straightforward [6] (note 1).

In 1857 Riemann proved the inequality $l(D) \geq \deg D - g + 1$, by analytic methods on a Riemann surface [7, 8]. Here $l(D)$ is the dimension of the space of functions with poles bounded by the divisor D, and g is the genus. Riemann surfaces are treated further in the chapter on unifying structures.

In 1865 Roch turned the inequality into the equality $l(D) - l(K - D) = \deg D - g + 1$, by adding the correction term $l(K - D)$, where K is the canonical divisor [7, 8]. A quick check: for $D = 0$ the formula reads $1 - g = 1 - g$ for $g = 0$ to 4, and $\deg K = 2g - 2$ gives -2, 0, 2, 4 and 6. The result rests directly on Riemann's inequality of 1857.

In 1873 Max Noether proved his AF + BG theorem, in Math. Ann. 6, 351–359 [9, 10, 11]. It says when a curve through the intersection of $f = 0$ and $g = 0$ can be written as $af + bg$, using local "Noetherian conditions" at the common points. It rests on no earlier link in this chain.

In 1874 Brill and Noether published a joint memoir in Math. Ann. 7, 269–370 [12, 13]. It gave algebraic-geometric proofs of results that Riemann and Clebsch–Gordan

had reached by transcendental means [13]. According to MacTutor, they gave the Riemann–Roch theorem its name [12]. The memoir rests on Roch's theorem of 1865.

The Italian school and the classification of surfaces

In 1891 Castelnuovo developed linear systems on surfaces, with the characteristic series and the adjoint system [14, 15, 16, 17]. In 1896 he gave his rationality criterion: a surface with $q = P_2 = 0$ is rational (note 2). The work rests on the memoir of Brill and Noether. MacTutor says so, and Hodge's obituary of 1953 calls it "a fusion" of Segre's geometry with "the famous memoir of Brill and Noether" [14, 16]. The problem had been posed by Clebsch and Max Noether.

Castelnuovo also proved the contractibility theorem: a smooth curve $E \cong P^1$ with self-intersection $E^2 = -1$ can be blown down [18, 19]. This is the basis of minimal models of surfaces. Its date is uncertain; one source places it around 1900 (note 3). No earlier link is named as its basis.

In December 1904 Enriques gave the "completeness theorem" for irregular surfaces, which MacTutor calls the key to the classification [20, 21]. According to Kleiman the construction is flawed, as Severi pointed out in 1921 [21]. Poincaré gave a rigorous analytic proof in 1910–11, and Grothendieck's Picard scheme settled it algebraically (note 4).

By 1914 Castelnuovo and Enriques had completed the birational classification of complex algebraic surfaces [14, 15, 22]. Their main tool was the Riemann–Roch theorem on surfaces [14]. Minimal models, with no (-1)-curves, are sorted by their plurigenera [15, 22]. A quick check with Noether's formula $\chi(O) = (K^2 + e)/12$: P^2 gives $(9 + 3)/12 = 1$, $P^1 \times P^1$ gives $(8 + 4)/12 = 1$, a K3 surface gives $(0 + 24)/12 = 2$, and an Enriques surface $(0 + 12)/12 = 1$. The classification rests on Castelnuovo's work of 1891 to about 1900, and, according to MacTutor, on Enriques' completeness theorem as its key [20].

Between 1892 and 1924 the Italian school gave several proofs of the resolution of singularities of surfaces: del Pezzo (1892), Levi (1899), Severi (1914), Chisini (1921) and Albanese (1924) [23, 24]. In 1935 Zariski showed that none of them was complete. For that reason none of them carries an arrow to later work.

Algebraic foundations

In 1931 F. K. Schmidt proved the Riemann–Roch theorem for curves over perfect fields, by a purely algebraic proof in function fields [8, 25]. According to Roquette, quoted by MacTutor, the proof "works for arbitrary perfect base fields" [25]. No earlier link is named as its basis.

In 1935 Walker gave the first rigorous resolution of singularities of surfaces over the complex numbers, replacing the Italian arguments [23, 26]. In 1936 Lefschetz wrote that Walker's proof, in the Annals of April 1935, was the only "certifiable" one [26]. It repairs the Italian proofs.

From 1935 into the 1940s Zariski laid algebraic foundations for the subject: generic points, valuations and normal varieties [24, 27]. His book *Algebraic Surfaces* (1935) began this work. He replaced the topological and analytic arguments of the Italians with valuation theory and ideal theory. The sources say that Zariski, van der Waerden

and Weil built the foundations together; that Zariski's proofs use van der Waerden's results is an inference (note 5).

Zariski proved resolution of singularities in characteristic 0 for surfaces in 1939 and for threefolds in 1944 [23, 24]. The method is local uniformisation of valuations. It rests on his own foundations and repairs the gap left by the Italian school. Hironaka's general result came 1964 − 1944 = 20 years later.

In 1946 Weil published *Foundations of Algebraic Geometry*, which set up abstract varieties and intersection theory over fields of any characteristic [28, 29, 30]. He needed these for his proof of the Riemann hypothesis for curves over finite fields, announced in 1940, proved in 1941, with details in 1948. The line from Weil to Deligne is followed in another chapter. The book builds on van der Waerden's algebra; no source says it uses specific results (note 5).

Sheaves, cohomology and Riemann–Roch in higher dimensions

Kodaira received the Fields Medal in 1954 for his work on harmonic integrals [22, 31, 32]. Using sheaf cohomology, he showed that Kähler varieties of a certain kind are Hodge manifolds, that is, projective [31]. In papers I–IV (1964–1968) he classified compact complex surfaces [15, 22]. He reworked the classification of Castelnuovo and Enriques and extended it to non-algebraic surfaces, which added the class VII surfaces. It rests on the classification of 1914, completed 1968 − 1914 = 54 years later.

In 1955 Serre published FAC, "Faisceaux algébriques cohérents", in Annals 61 [33, 34, 37]. It introduced cohomology with values in a sheaf, and coherent algebraic sheaves on varieties and on projective space [34]. FAC proved that the structure sheaf of a variety is coherent. In 1956 GAGA followed, in Ann. Inst. Fourier 6, 1–42 [35, 36]. On a projective variety every coherent analytic sheaf comes from a unique algebraic one, and "cette correspondance préserve les groupes de cohomologie" [36]. Neither rests on an earlier link in this chain.

In 1954 Hirzebruch proved the Riemann–Roch theorem in n dimensions, using the Todd class, in a note in PNAS [7, 8, 38, 39]. In 1957 Grothendieck proved a version for proper morphisms, which introduced the K-groups K(X) [40, 41]. The details were published by Borel and Serre in 1958. A quick check of Hirzebruch's formula on P^3: $\chi(O(k)) = (k+1)(k+2)(k+3)/6$, the binomial coefficient $C(k+3, 3)$, for symbolic k. Both theorems generalise Roch's theorem of 1865; no source says the proofs use it.

In the 1960s Grothendieck rebuilt the subject on schemes, in EGA and SGA [27, 37, 42]. That chain is treated elsewhere and not repeated here. Serre's sheaf theory influenced it; nLab says FAC's "sheaf-theoretic setup has certainly influenced a lot Grothendieck" [37].

Resolution of singularities after Zariski

In 1964 Hironaka proved resolution of singularities in any dimension in characteristic 0 [43, 44]. He used new algebraic tools and adapted old ones [44]. The IMU says he "extended" Zariski's work and MacTutor that he "generalised" it, so the proof builds on Zariski's results of 1939 and 1944; neither says it uses them (note 6).

In positive characteristic the line runs separately [23, 45, 46, 47]. Abhyankar resolved surfaces in 1956 and threefolds in characteristic greater than 5, that is $p \geq 7$, in 1966 (note 7). Cossart and Piltant extended this to threefolds in every positive characteristic in 2008–09. Dimension 4 and above is open. There is no arrow from Hironaka, since the methods are different. Cossart and Piltant extended Abhyankar's work, according to Wikipedia, Kyoto and their papers in J. Algebra (2008 and 2009), as reviewed by Abramovich [23, 45, 46].

Between 1991 and 2007 Bierstone and Milman, Villamayor, Encinas and Hauser, Włodarczyk, and Kollár simplified resolution in characteristic 0 [23, 48]. Their resolutions are strong and functorial with respect to smooth morphisms. According to Kollár, the different methods end in very similar resolution procedures [48]. They rest on Hironaka's work: all later results have "their seeds in [Hir64]", with much improved methods [48].

Surfaces in characteristic p and moduli of curves

In 1969 Mumford, and in 1976–77 Bombieri and Mumford, classified surfaces in characteristic p, extending Enriques' theory [22, 49]. In characteristics 2 and 3 new types appear: non-classical Enriques surfaces, quasi-hyperelliptic surfaces and quasi-elliptic surfaces [22]. The work rests on the classification of Castelnuovo and Enriques of 1914. It took $1977 - 1969 = 8$ years.

In 1969 Deligne and Mumford proved that the moduli space of curves of genus g is irreducible over every algebraically closed field, and so in all characteristics [50, 51, 52, 53]. They used stable curves and their deformation theory, stable reduction of abelian varieties applied through the Jacobian, and algebraic stacks. A quick check: $\dim M_g = 3g - 3$ gives 3, 6, 9, 12 and 15 for $g = 2$ to 6. The proof rests on the classical irreducibility over C (Enriques–Chisini), by reduction to characteristic zero. The paper's introduction says so, and arXiv 2005.04134 describes "reduction to characteristic zero, and Teichmüller theory over C" [51, 53] (note 8).

Mori's programme and minimal models

In 1979 Mori proved Hartshorne's conjecture of 1970: a smooth projective variety with ample tangent bundle is projective space P^n [54, 55, 56]. He produced rational curves by reduction to characteristic p and a technique called bend and break [56]. The paper is in Ann. Math. 110 (1979) (note 9). The conjecture was proved $1979 - 1970 = 9$ years after it was posed.

In 1982 Mori proved the cone theorem and the existence of extremal contractions for smooth threefolds [58, 59, 60]. Rational curves of bounded degree generate the K-negative part of the cone of curves. The result rests on Mori's work of 1979: Debarre shows that the bend-and-break lemma of the 1979 paper drives the proof [60]. It generalises Castelnuovo's contraction theorem, as an influence.

From about 1980 to 1992 Kawamata, Reid, Shokurov and Kollár extended the cone and contraction theorems to singular varieties [15, 59]. They used the "X-method" [61], built on the vanishing, non-vanishing, base-point-free and rationality theorems. A modern treatment proves these in that order [62], but that this was the historical route is an inference. There is no arrow from Mori's work of 1982 as a proof step:

Fujino calls the X-method “the completely different method” [61]. Mori’s work was an influence.

In 1988 Mori proved the existence of flips in dimension 3, so minimal models of threefolds exist [15, 63]. The paper appeared in JAMS 1, 117–253; Kollár calls it “a very difficult paper” [15]. The sources consulted do not describe the method. It builds on Mori’s cone and contraction theorems of 1982, which the paper cites; Kollár describes a flip as the step after a contraction [15].

From 1992 to 2007 Shokurov, and then Hacon and McKernan, developed pl-flips and an inductive approach [18, 62]. Hacon and McKernan reduced flips in dimension n to the minimal model programme in dimension $n - 1$, using restricted algebras and a lifting lemma. No earlier link is named as its basis.

In 2006 (published 2010) Birkar, Cascini, Hacon and McKernan proved that minimal models exist for varieties of log general type, and that the canonical ring is finitely generated [18, 62, 64, 65]. The proof is an induction on dimension, combining existence and finiteness of models with special termination of flips with scaling [62]. It rests on the work of Shokurov and of Hacon and McKernan [18, 62]. It came $2010 - 1988 = 22$ years after Mori’s flips.

In 2016 Birkar proved the BAB conjecture: Fano varieties of fixed dimension with ε-log canonical singularities form a bounded family [66, 67, 68, 69, 70]. He also proved boundedness of complements. The papers appeared in Annals 190 (2019) and 193 (2021), and he received the Fields Medal in 2018. The proof uses bounds on log canonical thresholds of linear systems on Fano varieties. It proves the BAB conjecture and rests on Shokurov’s conjecture on complements [69, 70]. The paper relies on “standard results of the minimal model program”; whether it uses the work of Birkar, Cascini, Hacon and McKernan specifically was not confirmed (note 10).

Arithmetic crossings and Hodge atoms

In 1983 Faltings proved Mordell’s conjecture, by “methods of arithmetic algebraic geometry” [28, 71, 72, 73]. The paper is in Invent. Math. 73, 349–366, and he received the Fields Medal in 1986. He is also an Abel Prize laureate of 2026 [72]. This link crosses to the chain on number theory (MSC 11).

On 7 August 2025 Katzarkov, Kontsevich, Pantev and Yu posted “Birational invariants from Hodge structures and quantum multiplication” (arXiv 2508.05105) [74, 75, 76]. As of 29 October 2026 it is still a preprint: version 2 is dated 6 March 2026 and there is no journal reference. It defines “Hodge atoms”, birational invariants built from Hodge structures and quantum multiplication, by spectral splitting of an F-bundle under the Euler field. It proves that a very general cubic fourfold is not rational. It also gives a new proof that birational Calabi–Yau manifolds have equal Hodge numbers. The invariants behave additively under blow-ups “in accordance with Iritani’s blowup theorem”, so the work builds on Iritani [74]. The IMSA conference on atoms of April 2026 listed a talk on “Iritani’s blow-up formula” [76]. (A quick check: 7 August 2025 to 29 October 2026 is 448 days.)

Two preprints of 2026 build on the atoms [76, 77, 78]. Guéré showed that rational smooth cubic fourfolds have primitive cohomology isomorphic to a twisted K3 Hodge structure [77]. Benedetti, Fay, Guéré, Manivel and Perrin gave an atomic irrationality criterion that needs no computation of atoms [78]. It applies to cubic, Gushel–

Mukai and Küchle (c5) fourfolds, and uses the monodromy equivariance of quantum multiplication and the irreducibility of the monodromy representation. Guéré's paper says it is "building on" the work of Katzarkov, Kontsevich, Pantev and Yu, and was also presented at the IMSA conference [76, 77]. The second paper "builds on the birational invariants introduced by" them [78].

Open problems, 28 October 2026

Several problems remain open, and none of them carries an arrow yet. The Hodge conjecture is unsolved [79, 80]. Termination of flips is open [15, 81]. Abundance is open [15]; a seminar at ECNU in 2025 reports only partial progress [82]. Resolution of singularities in characteristic p, in dimension 4 and above, is open [23]; Kyoto notes that Hironaka's announced proof is unconfirmed [45].

Notes

1. Who repaired Bézout's gap is not settled by the sources. The Leiden thesis names Halphen (1873) for multiplicities [5]; Wikipedia names Macaulay's proof by the U-resultant and Serre (1958) [3]; the CWI survey attributes the general theorem to van der Waerden (Math. Ann. 99, 1928) [6]. The Leiden thesis also dates Bézout's proof to 1764, against 1779 in MacTutor, the Encyclopedia of Mathematics and Wikipedia [1, 2, 3, 5].
2. Castelnuovo's rationality criterion dates from 1896, according to Kollár, Dolgachev and Hodge's obituary [15, 16, 17].
3. The date of about 1900 for the contractibility theorem rests on one source; no other source consulted gives a year [18, 19].
4. Enriques' completeness construction is dated December 1904, not 1905, and it was flawed [21]. Its role as "the key" to the classification rests on MacTutor alone [20].
5. That Zariski's and Weil's proofs use van der Waerden's results is an inference; the sources say only that the three built the foundations together [27].
6. That Hironaka's proof uses Zariski's results is an inference from "extended" and "generalised" [43, 44].
7. Abhyankar's bound is given as characteristic greater than 6 in some accounts and greater than 5 in others [23, 47]; for primes the two conditions are the same, $p \geq 7$.
8. Fulton's work for $p > 2g + 1$ and Severi's incomplete argument, as inputs to Deligne and Mumford, rest on the paper's own text only [51]. Wikipedia's page on moduli of curves has garbled bounds for Harris–Mumford and Eisenbud–Harris, so those results are left out [52].
9. MacTutor dates Mori's proof to 1978, but the paper appeared in Ann. Math. 110 (1979), so the chain uses 1979 [54]. MacTutor says "Hartshorne conjecture" and arXiv math/9607217 says "Hartshorne–Frankel conjecture" [54, 55]. Wikipedia explains both names: Frankel's differential-geometric version was resolved in 1979 by Mori and by Siu and Yau (published 1980), Hartshorne's algebraic version by Mori alone [57].
10. Birkar's paper relies on "standard results of the minimal model program", but its reference list was not readable, so its use of Birkar–Cascini–Hacon–McKernan is not confirmed [66].

11. Only the preprints themselves, and the IMSA conference programme, state the dependencies of the atom papers on Iritani and on Katzarkov, Kontsevich, Pantev and Yu [74, 76, 77, 78]. That abundance was still open in 2026 rests on Kollár (2014) and one seminar report [15, 82].
12. Settled dates: Hirzebruch–Riemann–Roch 1954, Faltings' proof 1983, the journal versions of Birkar's papers (Annals 190, 2019, and 193, 2021), GAGA 1956. The MSC title of the field is "Algebraic geometry" [83, 84]. The Plücker formulas, the 27 lines on a cubic surface, Zariski's main theorem and the Zariski topology are outside this chain.

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Category theory and homological algebra

Category theory studies mathematical structures through the maps between them, and homological algebra studies the algebraic invariants that such maps carry. The chain starts in algebraic topology. Eilenberg and Mac Lane's 1942 paper on Hom and Ext led to their 1945 definition of categories, with Emmy Noether's structural view as background only.

Three lines follow from there. The first is homological algebra: Cartan and Eilenberg in 1956, Grothendieck's Tôhoku paper in 1957, the Freyd–Mitchell embedding of 1964, and derived categories up to Spaltenstein in 1988. The second is structure and logic: Yoneda, Kan's adjoint functors of 1958, coherence, and toposes up to Lawvere and Tierney in 1969. The third is higher categories: Quillen in 1967, Boardman and Vogt in 1973, Joyal in 2002 and Lurie in 2009.

Two points are still open in 2026. The cobordism hypothesis has no published complete proof, and the Lean formalisation of ∞-categories is in progress; its first paper is refereed (ITP 2025; checked 29 October 2026). Buchsbaum's 1955 work and the Tôhoku paper were independent, parallel discoveries. Whether the Tôhoku paper rests on Cartan–Eilenberg is disputed (note 1).

From Noether to the definition of a category (1920s–1955)

In the 1920s Emmy Noether taught a structural view of algebra. A structure is studied through its structure-preserving maps, with subgroups, quotients and the homomorphism theorem as the basic notions [1, 2]. This is background, not a theorem. McLarty describes it as an anticipation of category theory, not a documented influence [2].

In 1942 Samuel Eilenberg and Saunders Mac Lane introduced Hom and Ext, according to MacTutor and Weibel "for the first time" [3, 5]. The same work introduced the terms "functor" and "natural isomorphism" [3]. Their specific functors and natural transformations were limited to groups [4]. (A quick check: $\mathrm{Ext}^1(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/n\,/\,m\mathbb{Z}/n$ has order gcd(m, n); for the pairs (4, 6), (5, 7), (12, 18) and (8, 8) this gives 2, 1, 6 and 8.)

In 1945 the same authors published "General theory of natural equivalences". It defined categories, functors and natural transformations, together with limits and colimits, so that natural transformations would make sense in general [4, 1]. The paper abstracts the results of 1942; Noether counts only as an anticipation. That was 81 years before 2026.

Also in 1945, Eilenberg and Norman Steenrod announced axioms for homology; the proofs followed in their book *Foundations of Algebraic Topology* (1952) [3, 6, 7]. Homology assigns groups to pairs of spaces and homomorphisms to maps, subject to axioms that include excision. For a given coefficient group the theory is unique on compact polyhedra. (A quick check: a triangle, which is a circle, has $b_0 = 1$ and $b_1 = 1$, since its boundary matrix has rank 2; and $\chi = V - E = 0 = b_0 - b_1$.) This work rests on Eilenberg–Mac Lane 1945. Chapter IV of the book presents "the ideas

and language of category and functor", and the Stanford Encyclopedia notes that Eilenberg and Steenrod defined categories there [7, 4]. The uniqueness claim is confirmed by the Encyclopedia of Mathematics (for compact polyhedra) and by the book's preface: "the axioms are categorical for homology theories on triangulable spaces" [6, 7].

The Yoneda lemma states Nat(y(c), X) ≅ X(c): a natural transformation out of a representable functor is fixed by where it sends the identity id_c [8]. Mac Lane dated it to 1954 and named it after an interview with Nobuo Yoneda at the Gare du Nord in Paris [8, 9]. (A quick check by brute force, on finite sets of size 1 and 2: Nat(Hom(2,−), Hom(1,−)) has 2 elements, as many as Hom(1,2), and Nat(Hom(2,−), Hom(2,−)) has 4, as many as Hom(2,2).) The lemma has no recorded predecessor in this chain. Yoneda's own 1954 paper contains his classification of Ext, not the lemma in its modern form [10] (note 3).

In 1955 David Buchsbaum published "Exact categories and duality" (Trans. AMS 80). It gave axioms for categories with kernels, cokernels and duality: abelian categories under another name [4, 11]. This work and Grothendieck's Tôhoku paper were independent, parallel discoveries, with Buchsbaum first (note 7).

Homological algebra and derived categories (1956–1988)

In 1956 Henri Cartan and Eilenberg published *Homological Algebra*; the manuscript had been finished in 1953 [3]. It treats derived functors systematically [5]. Categories are assumed in it, not studied [4].

In 1957 Alexander Grothendieck published the Tôhoku paper [12, 13]. It introduced additive and abelian categories and the AB axioms; AB5 guarantees enough injectives. It also gave derived functors and the Grothendieck spectral sequence. The aim was to give modules and sheaves one common homological algebra. Whether it rests on Cartan–Eilenberg is disputed: Weibel says Grothendieck "proceeded to generalize" their treatment of derived functors, while nLab says he worked independently, without the manuscript [5, 13] (note 1).

In the early 1960s Grothendieck and his student Jean-Louis Verdier created derived and triangulated categories. Complexes are localised at the quasi-isomorphisms, so that these maps become invertible [14, 15, 5]. The motive was coherent duality [14, 16]. Verdier's "Catégories dérivées, état 0" dates from 1963. His thesis was defended on 14 June 1967 and printed only in Astérisque 239 in 1996, 29 years later [17, 16] (note 4). This line builds on the abelian categories of the Tôhoku paper.

In 1964 Peter Freyd and Barry Mitchell proved the embedding theorem. Every small abelian category has a full, faithful and exact functor into R-Mod, the modules over some ring R [18, 19, 20]. The theorem is named after Freyd's book *Abelian Categories* (1964) and Mitchell's paper in Amer. J. Math. 86 (1964). In the usual proof one embeds the category into its left exact functors by the Yoneda embedding, finds an injective cogenerator there, and takes R to be its endomorphism ring [19] (note 9). The proof uses a Grothendieck category with an injective cogenerator, so it builds on the Tôhoku framework [19]. Weibel adds that it built on the 1960 embeddings into abelian groups by Lubkin, Heron and Freyd [5]. The result is 62 years old in 2026.

In 1988 Nicolas Spaltenstein extended derived categories to unbounded complexes (Compositio 65, 121–154) [21]. He introduced K-injective and K-projective resolutions

[15]. In Weibel's words, he showed that every unbounded complex is quasi-isomorphic to a "fibrant" one [5]. This rests on Verdier's derived categories.

Adjoints, coherence and toposes (1958–1972)

In 1958 Daniel Kan published adjoint functors (Trans. AMS 87, 294–329); he had defined them in 1956 [22]. An adjunction is a natural bijection Hom(FA, B) ≅ Hom(A, GB). Kan showed that inverse limits are right adjoints of the constant-diagram functor (his Theorem 8.6). Adjoints subsume limits and colimits [4]. The paper, received on 20 September 1956, cites Eilenberg–Mac Lane 1945 for the definitions of category, functor and natural transformation, 13 years earlier [22] (note 8).

Around 1963 Grothendieck introduced what are now called Grothendieck toposes: categories of sheaves on a site. He needed them for cohomology theories in algebraic geometry [23, 24].

In 1963 Mac Lane proved the coherence theorem for monoidal categories (Rice University Studies 49) [25, 26]. Every formal diagram built from associators and unitors commutes, given the pentagon and four unit conditions. As a consequence every monoidal category is equivalent to a strict one. (A quick check: the word abcd has 5 bracketings, joined by 5 associator edges, which form the pentagon; the Catalan numbers for 2 to 6 objects are 1, 2, 5, 14 and 42.) In 1964 Max Kelly simplified the theorem (J. Algebra 1, 397–402). He showed three of Mac Lane's four unit conditions redundant, leaving one condition that relates units and associativity [27, 28] (note 5).

From 1963 to 1969 William Lawvere axiomatised sets and categories through arrows. His work includes functorial semantics (thesis, 1963), the elementary theory of the category of sets, ETCS (1964), the category of categories (1966) and "Adjointness in Foundations" (1969) [4, 29].

In 1967 Jean Bénabou introduced bicategories (LNM 47) [30, 31]. In a bicategory, composition is associative only up to coherent 2-cells. A bicategory with one object is the same as a monoidal category. The strictification theorem, that every bicategory is biequivalent to a 2-category, is not Bénabou's: Leinster credits it to Street and to Gordon, Power and Street [32]. Mac Lane's monoidal coherence carries over to bicategories [31, 32]. Since that bicategorical coherence theorem came later, the link to Mac Lane is a relation of ideas rather than a step in Bénabou's 1967 paper (note 6).

In 1969 Lawvere and Myles Tierney defined the elementary topos; it was printed in 1970 [29, 23]. An elementary topos is a cartesian closed category with finite limits and a subobject classifier. First-order axioms thus replace Grothendieck's site. The notion generalises Grothendieck toposes and was built from SGA 4, according to the Lawvere Archives [33]. It also rests on Lawvere's work on foundations: the ICM paper cites "Adjointness in Foundations" (1969) [29, 4]. And it uses Kan's adjoints, 11 years after Kan, through cartesian closure: the ICM paper defines exponentials as right adjoints to the product, and the Stanford Encyclopedia credits adjoints to Kan [29, 4]. The date is discussed in note 2.

In 1971 Tierney proved the independence of the continuum hypothesis with sheaves inside topos theory; it was printed in 1972 in LNM 274 [33, 4]. The Stanford Encyclopedia speaks of "geometric proofs of the independence of the continuum hypothesis" [4]. This rests on Lawvere–Tierney 1969: Tierney had worked out the

logic in the seminar of 1969–70 [33].

Also in 1971 Anders Kock and Gavin Wraith published *Elementary Toposes* (Aarhus Lecture Notes Series 30, September 1971) [24, 34, 35]. It is an early exposition, written from lectures of 1970–71. Its Chapters 1 and 3 follow Tierney's Halifax lectures of 1969–70, so it rests on Lawvere–Tierney [34].

Higher categories (1967–2025)

In 1967 Daniel Quillen introduced model categories (*Homotopical Algebra*, LNM 43) [36, 37]. A model category has weak equivalences, fibrations and cofibrations, and its homotopy category is a localisation. Spaces, simplicial sets and chain complexes are the examples. Quillen generalised homological algebra by analogy [37, 38]; neither source names Cartan–Eilenberg or the Tôhoku paper as used in a proof (note 10). The theory is 59 years old in 2026.

In 1973 Michael Boardman and Rainer Vogt introduced weak Kan complexes, later called quasi-categories. These are simplicial sets in which only inner horns need fillers [39, 40]. Joyal took them up 29 years later.

In 1995 John Baez and James Dolan stated the cobordism hypothesis (J. Math. Phys. 36) [41, 42]. It is a conjecture: fully extended topological quantum field theories correspond to fully dualizable objects. It does not rest on an earlier proof in this chain.

In 2002 André Joyal published "Quasi-categories and Kan complexes" (JPAA 175, 207–222) [43]. It shows that a quasi-category in which all morphisms are invertible is a Kan complex [44] (note 9). Joyal proposed quasi-categories as a basis for category theory and gave the Joyal model structure [39]. This rests on Boardman–Vogt 1973: Joyal worked on quasi-categories after their invention by Boardman and Vogt [39, 40].

In 2009 Jacob Lurie published *Higher Topos Theory*, which had appeared on arXiv from 2006 to 2008 [45, 46, 47]. It develops ∞-categorical limits, adjoints, presheaves, the Yoneda lemma and ∞-topoi, all built on quasi-categories. It rests on Joyal 2002, seven years earlier; the book uses "Joyal's theory of quasi-categories" [45]. Behind that stands Boardman–Vogt 1973. It generalises Grothendieck toposes [47, 46]. It also generalises the Yoneda lemma: nLab cites Proposition 5.1.3.1 of the book [46, 48].

Also in 2009 Lurie wrote "On the Classification of Topological Field Theories" [49]. It gives an informal sketch of a proof of the cobordism hypothesis, using n-fold complete Segal spaces. It rests on Baez–Dolan 1995, a conjecture that is sketched but not fully proved. A talk abstract from September 2025, 16 years later, states that no complete rigorous proof has been published [50].

In 2013 the Univalent Foundations Program published the *Homotopy Type Theory* book [51, 4]. Types are read as homotopy types, with Voevodsky's univalence axiom and higher inductive types. A link from topos theory is not documented, so it stands as an independent line here.

In 2021–22 Daniel Grady and Dmitri Pavlov proposed a geometric cobordism hypothesis [52, 41]. It generalises the Baez–Dolan and Hopkins–Lurie statement to bordisms with geometric structure, by means of a locality theorem. It is a preprint

(arXiv 2111.01095, version 3 of 19 June 2022), with no journal reference as of 29 October 2026. It rests on Baez–Dolan and Lurie 2009 (note 7).

In 2024–25 Mario Carneiro and Emily Riehl began to formalise ∞-category theory in Lean through ∞-cosmoi [53, 54]. Their first paper, "Formalizing colimits in Cat", proves that the homotopy category functor is left adjoint to the nerve, that the nerve is fully faithful, and that Cat is cocomplete [53]. It appeared at ITP 2025 (LIPIcs 352, 22 September 2025; arXiv 2503.20704) [55]. (A quick check: from the project launch in September 2024 to the arXiv paper in March 2025 is 6 months.) The wider project is still in progress. Joyal's quasi-categories are background here, as the prototypical ∞-cosmos, but not a proof step [56]. The formalised results concern the 1-category Cat, the project is axiomatic, and in Riehl's words "proving that quasi-categories define an ∞-cosmos will be hard" [53, 54]. A link to *Higher Topos Theory* rests only on overlapping authors (note 11).

Notes

1. Whether the Tôhoku paper builds on Cartan–Eilenberg is unsettled. Weibel says Grothendieck "proceeded to generalize" their treatment of derived functors [5]. nLab says he had no access to the book and developed overlapping constructions independently [13]. The Stanford Encyclopedia, cited earlier for "generalisation", says neither [4]. The arrow stays dashed.
2. The elementary topos is dated 1969 by the Encyclopedia of Mathematics (which dates its reference 1971), 1969–70 by the Lawvere Archives and 1972 by the Stanford Encyclopedia [23, 33, 4]. The primary source settles it: introduced in 1969 and printed in the 1970 ICM Actes [29]. 1971 is when that volume appeared, and 1972 matches LNM 274.
3. The Yoneda date 1954 is Mac Lane's (nLab, citing *Categories for the Working Mathematician*), and arXiv 1404.6240 gives it too [8, 9]. Yoneda's 1954 paper contains his Ext classification rather than the lemma in modern form [10].
4. Verdier defended his thesis on 14 June 1967 (Illusie's preface to Astérisque 239) [16]. Weibel's "1963 thesis" means the état 0 text [5].
5. Kelly 1964 simplified Mac Lane's coherence conditions: three of the four unit axioms are redundant [27]. For 1963 one should therefore speak of the pentagon and four unit conditions, not "pentagon and triangle".
6. The strictification of bicategories is not Bénabou's 1967 result; Leinster credits Street and Gordon–Power–Street [32]. The link from Mac Lane's coherence to Bénabou is therefore one of ideas. Kock–Wraith's notes claim no priority, so they are called an early exposition, not the first systematic one [34].
7. Buchsbaum (1955) and Grothendieck introduced abelian categories independently, with Buchsbaum first; the Stanford Encyclopedia and nLab both allow this [4, 11]. Grady and Pavlov claim a geometric generalisation in an unrefereed preprint, while Natori (September 2025) says no complete rigorous proof of the cobordism hypothesis has been published [52, 50]. Both can hold, so the status stays open.
8. Only Kan's paper itself states that it rests on Eilenberg–Mac Lane 1945 [22].
9. Only Wikipedia gives the proof idea of the Freyd–Mitchell theorem and its use of a Grothendieck category [19]. Only the Oslo course notes state Joyal's result in the form "all morphisms invertible implies Kan" [44].
10. Quillen's link to homological algebra is one of analogy, not of a result used in a proof [37, 38].

11. The link from *Higher Topos Theory* to Carneiro–Riehl rests on overlapping authors only. Not covered in this chapter, for lack of sources: operads (May 1972, Boardman–Vogt), Dwyer–Kan localisation, Rezk's complete Segal spaces (2001), the Joyal–Tierney and Bergner comparisons, t-structures (1982), Bondal–Kapranov and Orlov, Kontsevich's homological mirror symmetry (1994) and stable ∞-categories. Lawvere's 1963 thesis is covered only in general terms. No refereed complete proof of the cobordism hypothesis, and no other 2026 result in this field, was found as of 29 October 2026.

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K-theory

K-theory attaches abelian groups to rings, schemes and spaces, built from vector bundles, projective modules and matrices. It starts in algebraic geometry: Grothendieck built K_0 in 1957 to prove his Riemann–Roch theorem. Atiyah and Hirzebruch carried it into topology, where Bott periodicity, Adams's vector-fields theorem and the index theorem used it at once.

The algebraic line runs from Bass's K_1 and Milnor's K_2 to Quillen's higher K-theory of 1972–73. It then splits in two. A motivic line runs from Merkurjev–Suslin in 1982 to Voevodsky's proof of the Bloch–Kato conjecture in 2011. A trace-method line runs from Dennis in 1976 to Nikolaus–Scholze in 2018. The newest results, the computation of $K(\mathbb{Z}/p^n)$ and the disproof of the telescope conjecture (Clay Research Award 2026), are both still preprints as of 29 October 2026. Borel's 1974 ranks stand on their own, independent of the rest of the chain.

Origins: Serre's problem, Grothendieck's K_0 and topology (1955–1968)

In 1955, in his paper FAC, Jean-Pierre Serre asked whether finitely generated projective modules over a polynomial ring $k[x_1, \ldots, x_n]$ are free [1, 2]. It was a problem, not a theorem, and it stood for 21 years, until Quillen and Suslin solved it in 1976.

In 1958 Serre gave a partial answer: such modules are stably free [1, 2, 3]. The result is Proposition 10 of his Séminaire Dubreil exposé dated 5 May 1958 [3] (note 3). The proof uses a finite free resolution, which Hilbert's syzygy theorem supplies; only the exposé itself describes this method (note 9).

In 1957 Alexander Grothendieck defined the group K_0 of coherent sheaves and vector bundles; the K stands for "Klasse" [4, 5]. It is the free abelian group on isomorphism classes, modulo the relations given by short exact sequences. He built it to state the Grothendieck–Riemann–Roch theorem for proper push-forward. (A quick check, as in the chapter on algebraic geometry: $\chi(\mathcal{O}(k))$ on $\mathbb{P}^3$ equals binomial$(k + 3, 3)$.) This generalises the Hirzebruch–Riemann–Roch theorem, a link that crosses over from algebraic geometry.

Between 1956 and 1959 Raoul Bott proved his periodicity theorem by Morse theory [6, 7, 8, 9]. The stable homotopy groups of the unitary group U repeat with period 2, those of O and Sp with period 8. The proof studies the loop space of $SU(2n)$, whose minimal geodesics form a Grassmannian [8, 9]. (A quick check: KU of the spheres runs Z, 0, Z, 0, …; the reduced KO of S^n equals $\pi_{n-1}(O)$, which for n = 0 to 7 is Z, Z/2, Z/2, 0, Z, 0, 0, 0, with period 8.) The theorem was found in 1956–57, announced in 1957 and published in full in 1959 (note 2).

In 1959 Michael Atiyah and Friedrich Hirzebruch introduced topological K-theory [10, 11]. They applied Grothendieck's construction to vector bundles on spaces. With Bott periodicity it becomes a periodic, multiplicative cohomology theory [10]. The

main paper followed in 1961, two years later (note 1). It rests on Grothendieck's K_0, transported to topology, and on Bott periodicity.

Frank Adams used these tools for two classical problems [12, 10, 13, 14]. In 1960 (Annals 72, July 1960) he settled the Hopf invariant one problem by secondary cohomology operations, not by K-theory. In 1962 (Annals 75, May 1962) he determined the number of vector fields on spheres by K-theory and his Adams operations. These give the upper bound $\rho(n) - 1$, where ρ is the Radon–Hurwitz number. (A quick check: $\rho(n) - 1$ for n = 2, 4, 8, 16, 32, 6 and 12 is 1, 3, 7, 8, 9, 1 and 3; up to n = 1024 only S^0, S^1, S^3 and S^7 are parallelisable.) The 1962 paper rests on topological K-theory; the years are taken from the original papers (note 3).

From 1962 to 1968 Hyman Bass and his collaborators built the lower algebraic K-groups. Bass and Stephen Schanuel defined $K_1(R) = GL(R)/E(R)$ in 1962 [4, 15]. The quotient GL/E goes back to Whitehead [4, 15, 16]. The fundamental theorem, with its map $K_1(R[t, t^{-1}]) \to K_0(R)$, first appeared in the paper of Bass, Alex Heller and Richard Swan on the Whitehead group of a polynomial extension (Publ. IHÉS 22, 1964) [17]. Bass extended it in work with M. Pavaman Murthy in 1967, and introduced negative K-groups in 1968, according to Wikipedia's account [4] (note 10). (A quick check: $K_1(F_q)$ is the unit group $F_q\hat{}\times$, of order $q - 1$, which matches Quillen's later formula $Z/(q - 1)$.)

In 1963 Atiyah and Isadore Singer announced the index theorem; it had arisen in 1962 [18, 11]. The analytic index of an elliptic operator equals its topological index. The topological index is defined in K-theory, by a push-forward to $K(TY) \cong Z$. In 1968, five years after the announcement, they gave a proof in K-theory that avoids cohomology and cobordism. (A quick check, in the Gauss–Bonnet case: $\chi = 2 - 2g$ gives 2, 0, −2, −4 and −6 for g = 0 to 4.) The proof rests on topological K-theory, and on Bott periodicity through the K-theory Thom isomorphism that builds the topological index [19]. Atiyah and Singer received the Abel Prize in 2004.

In 1967 Bass, John Milnor and Serre solved the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$) (Publ. IHÉS 33) [20, 21]. Their method links number theory to K-theory. It rests on Bass's K_1: they prove $SK_1 = 0$ for rings of integers, so that $K_1 \cong R\hat{}\times$ (Weibel's K-book, citing [BMS, 4.3]) [15]. The result is 59 years old in 2026.

From K_2 to Quillen's higher K-theory (1967–1976)

In the spring of 1967 Milnor defined $K_2(R)$ as the kernel of the map from the Steinberg group St(R) onto E(R) [4, 22]. The kernel is central in the Steinberg group [22]. This rests on K_1, through the Steinberg group.

In 1968–69 Hideya Matsumoto gave a presentation of K_2 of a field by symbols {a, b} [4, 22]. The symbols are bilinear and satisfy the Steinberg relation $\{a, 1 - a\} = 1$. His method is not described in the sources consulted. (A quick check: the Steinberg relation holds for the Hilbert symbol; see Merkurjev–Suslin below.) It rests on Milnor's K_2; Weibel's K-book takes Matsumoto's presentation from Milnor's book [22, 16].

In 1970–71 Milnor defined what is now called Milnor K-theory, $K\hat{}M_n(F)$ (Inventiones 9, received 10 February 1970) [4, 23]. It is the tensor algebra on $F\hat{}\times$ modulo the elements $a \otimes (1 - a)$. Milnor called the definition "purely ad hoc" for $n \geq 3$. He proved that the map to the Witt ring is bijective for $n \leq 2$. He also asked whether $K_nF/2 \cong H\hat{}n(G_F; Z/2)$. Milnor says his definition of K_2 was "motivated by work of R.

Steinberg, C. Moore, and H. Matsumoto" [23]. So it builds on Matsumoto's symbols as motivation rather than as a proof step. Milnor's K_2 of 1967 and Milnor K-theory of 1970 are two separate events, often merged (note 4).

In 1971 Daniel Quillen proved the Adams conjecture on the stable J-homomorphism (Topology 10), which Adams had posed eight years earlier in Topology 2 (1963) [7, 24]. MacTutor describes two approaches. One used algebraic geometry and was completed by his student; the other used the modular representation theory of groups, and Quillen completed it himself [25]. The 1971 proof takes a different route from Quillen's 1968 sketch by étale homotopy, which Friedlander completed [24, 26]. Only MacTutor states that the 1971 method is modular representation theory (note 9).

In 1972–73 Quillen created higher algebraic K-theory [25, 27, 28]. He defined $K_n(R) = \pi_n(BGL(R)^+)$ by the plus construction, and gave the Q-construction for exact categories; he proved that the two agree ("+ = Q"). He also computed the K-groups of finite fields: $K_{2i-1}(F_q) = Z/(q^i - 1)$ and $K_{2i}(F_q) = 0$ (Annals 96, 1972) [28]. (A quick check: $K_n(F_2)$ for n = 1 to 7 is Z/1, 0, Z/3, 0, Z/7, 0, Z/15, and $K_n(F_3)$ is Z/2, 0, Z/8, 0, Z/26, 0, Z/80.) This generalises Grothendieck's K_0 of 15 years earlier by homotopy methods. Quillen received the Fields Medal in 1978 [27].

In 1974 Armand Borel computed the stable real cohomology of arithmetic groups (Ann. ENS 7) [29, 4]. This gives the ranks of $K_n(Z)$, and of the K-groups of rings of integers, modulo torsion. Rognes and Weibel cite it as Borel's calculation of the rational ranks [30]. It does not rest on an earlier link in this chain.

Also in 1974 Spencer Bloch proved his formula $H^2(X, K_2) \cong CH^2(X)$, in "$K_2$ of algebraic cycles" [4, 16]. The Gersten–Quillen conjecture of the 1970s leads to Bloch's formula for all regular varieties, through the Gersten resolution of the K-sheaf [16]. Both rest on Quillen's K-sheaves [16], and on the Brown–Gersten–Quillen spectral sequence [31]. The year 1974 is given only by Wikipedia's reference list (note 9).

Quillen, following Stephen Lichtenbaum, formulated the Quillen–Lichtenbaum conjecture, dated 1975 by Wikipedia [4] (note 10). It predicts a spectral sequence from étale cohomology to K-theory. It rests on Quillen's K-theory: Rognes and Weibel call it Quillen's conjecture of a spectral sequence of Atiyah–Hirzebruch type, starting at étale cohomology [30].

In January 1976 Quillen and Andrei Suslin solved Serre's problem independently (Invent. Math. 36, 167–171; Doklady 229, 1063–1066) [32, 33, 34]. Projective modules over polynomial rings over a field, or over a principal ideal domain, are free. Quillen's proof builds on Horrocks's theorem of 1964. Dicks builds "an idempotent-matrix-based version of Quillen's proof" by "translating most of Paul Roberts' geometry-free proof of Horrocks' theorem" [33]. Lam's book pairs a chapter on Horrocks' theorem with one on Quillen's method [34]. (A quick check: the unimodular row $(x, 1 - x)$ completes to the matrix $[[x, 1 - x], [-1, 1]]$, whose determinant is 1.) The result closes Serre's problem of 1955.

The trace-method line (1976–2018)

In 1976 Keith Dennis defined the Dennis trace, a map from K-theory to Hochschild homology [4, 35]. It comes from the trace of matrices, applied on BGL. It rests on Quillen's K-theory; Hesselholt cites Dennis's 1976 lecture [35].

In the mid-1980s Marcel Bökstedt introduced topological Hochschild homology, THH:

Hochschild homology taken over the sphere spectrum [4, 35]. Bökstedt's trace to THH refines the Dennis trace (Hesselholt, citing Bökstedt's 1985 preprint) [35].

In 1993 Bökstedt, Wu-Chung Hsiang and Ib Madsen introduced topological cyclic homology, TC, and the cyclotomic trace [4, 36]. TC refines cyclic homology and approximates K-theory. It rests on Bökstedt's THH and on Quillen's K-theory.

In 1996 Bjørn Dundas, Tom Goodwillie and Randy McCarthy showed that relative K-theory equals relative TC for nilpotent extensions [4, 35, 37]. Hesselholt credits the statement to McCarthy (Acta Math. 1997) (note 5). A book followed in 2013 [37]. The result rests on the TC of Bökstedt, Hsiang and Madsen.

In 2017 Thomas Nikolaus and Peter Scholze gave new foundations for TC, published in Acta Math. 221 in 2018 [36, 38, 37]. They treat cyclotomic spectra through Tate constructions and give a simple formula for TC of bounded-below spectra. (A quick check: Bökstedt–Hsiang–Madsen 1993 to arXiv 2017 is 24 years, and arXiv 2017 to Acta 2018 is one year.) This reworks the TC of Bökstedt, Hsiang and Madsen: the two agree after p-completion when THH is bounded below (Rognes, citing Nikolaus–Scholze, Theorem II.3.8) [37]. It also builds on Dundas–Goodwillie–McCarthy, though Rognes indicates this only by a section title (note 9).

Foundations of the 1980s and 1990s, and the motivic line (1982–2011)

In 1982 Alexander Merkurjev and Suslin proved that the norm residue map $K_2(F)/n \to H^2(F, \mu_n^{\otimes 2})$ is bijective [39, 40, 31]. As a consequence, when F contains the n-th roots of unity, central simple algebras of exponent n are products of cyclic algebras. The proof uses the K-cohomology of Severi–Brauer varieties. (A quick check by exhaustive computation: the Hilbert symbol on Q_p satisfies $(a, 1-a)_p = 1$ for p = 3, 5, 7, 11 and all integers $0 < |a| \le 40$ with $a \neq 1$; it is bimultiplicative on nonzero integers in $[-12, 12]^3$; and $(3, 5)_5 = -1$, $(2, 3)_3 = -1$, $(-1, -1)_3 = 1$.) It settles the degree-2 case of Milnor's question for every n; Merkurjev had proved the case n = 2 shortly before (note 4). It rests on Gersten's work and the K-cohomology of Severi–Brauer varieties, with the Brown–Gersten–Quillen spectral sequence and Bloch's formula [31].

In 1985 Friedhelm Waldhausen introduced Waldhausen categories and the S•-construction (LNM 1126) [4, 16, 35]. They give the K-theory of categories with cofibrations and weak equivalences. Only Wikipedia's reference list gives the year (note 9).

The degree-3 case of the norm residue map for l = 2 was done independently by Markus Rost (1986) and by Merkurjev and Suslin [40, 41]. Merkurjev and Suslin's paper "The norm residue homomorphism of degree three" appeared in Izvestiya in 1990, and in English translation in 1991 [42]. It proves that $K_3^M(F)/2^n \to H^3(F, \mu_{2n}^{\otimes 3})$ is bijective, through an analogue of Hilbert's Theorem 90 for K_3^M [42]. It builds on the degree-2 theorem of 1982, according to Wikipedia (note 10). Rost's method was not examined.

In 1990 Robert Thomason and Thomas Trobaugh built the K-theory of schemes from perfect complexes, in Waldhausen's framework [4, 43]. They proved localisation and Mayer–Vietoris theorems. The work rests on Waldhausen 1985, together with derived categories: the introduction credits Waldhausen's K-theory of complexes with weak equivalences [43].

Vladimir Voevodsky proved the Milnor conjecture, K^M_n(F)/2 ≅ H^n_ét(F, Z/2) [44, 4, 41]. His preprints date from 1995–96, and the completed proof was printed in 2003 (Publ. IHÉS 98), 33 years after Milnor's question (note 6). The proof uses motivic cohomology, $\mathbb{A}^1$-homotopy theory and motivic Steenrod operations. It proves Milnor's conjecture. The degree-3 case of Merkurjev–Suslin and Rost is background, not a step [41]. Suslin's modification of motivic cohomology is, in Soulé's word, crucial [41]. Voevodsky received the Fields Medal in 2002.

Voevodsky then proved the Bloch–Kato conjecture, the norm residue isomorphism, for all primes l, with contributions by Rost and Charles Weibel [45, 44]. The proof uses motivic cohomology with Z/l coefficients and Rost motives. It appeared in Annals 174 in 2011, received on 4 January 2009, two years earlier [45]. It extends the Milnor conjecture to odd primes and rests on Rost's norm varieties.

Recent results (2022–2026)

In 2022–24 Benjamin Antieau, Achim Krause and Nikolaus computed the K-theory of $\mathbb{Z}/p^n$, and more generally of O_K/I, from prismatic (syntomic) cohomology [46, 47]. The even K-groups vanish in high degrees, the orders of the odd ones in high degrees are determined, and an algorithm computes individual groups. It is a preprint (arXiv 2405.04329, version 1 only, no journal reference) as of 29 October 2026. (A quick check: from 7 May 2024 to 29 October 2026 is 29 months.) It belongs to the trace-method line, as work of the same authors; its abstract names prismatic and syntomic cohomology but not Nikolaus–Scholze (note 7).

In 2023 Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank disproved the telescope conjecture at every prime and every height ≥ 2 [48, 49, 50]. They showed that the T(n+1)-localised algebraic K-theory of BP⟨n⟩^hZ is not K(n+1)-local. It is a preprint (arXiv 2310.17459, version 1 of 26 October 2023, no journal reference) as of 29 October 2026. It received the Clay Research Award 2026 (IHES, 27 April 2026) [49]. (A quick check: from October 2023 to April 2026 is 30 months.) It rests on the algebraic K-theory of ring spectra [48, 50]. The Oberwolfach Reports describe a line of work on localisations and algebraic K-theory, "culminating in the recent disproof" [50]. No source states that it uses TC, so there is no link from the trace methods (note 7).

Notes

1. Landsman dates the Atiyah–Hirzebruch groups to 1959 [11]. The 1961 date of the main paper is not given in the sources consulted (Wikipedia's page on topological K-theory gives no year).
2. Bott periodicity is dated 1956 (Morse theory of Lie groups), 1957 and 1959 by Wikipedia, 1957 by the Harvard biography, by the 1957 PNAS note in nLab, and by the 1959 Annals paper in the Encyclopedia of Mathematics [7, 9, 6, 8]. In short: found 1956–57, announced 1957, published in full 1959. The index theorem arose in 1962 (Landsman) and was announced in 1963 (Wikipedia), which is compatible [11, 18].
3. Serre's stably-free result is in his Séminaire Dubreil exposé of 5 May 1958; Wikipedia gives 1957 [3, 1]. Adams's 1960 Hopf-invariant-one paper uses secondary cohomology operations, not K-theory; only the 1962 vector-fields paper uses K-theory [13, 14].

4. Merkurjev–Suslin is dated 1982. Merkurjev's case n = 2 is dated 1981 by Wikipedia and 1983 by Soulé [40, 41]. Milnor's K_2 (spring 1967) and Milnor K-theory (1970–71) are two events, often merged, not a conflict.
5. Hesselholt credits the nilpotent-extension theorem to McCarthy (Acta 1997), Wikipedia to Dundas, Goodwillie and McCarthy (1996) [35, 4]. Both can hold.
6. Voevodsky's Milnor-conjecture preprints date from 1995–96; the completed proof was printed in 2003, the date used here.
7. No source consulted states that the telescope disproof uses TC, or that the $K(\mathbb{Z}/p^n)$ work uses Nikolaus–Scholze. The link from trace methods to Antieau–Krause–Nikolaus rests only on the overlap of authors. No journal publication of Antieau–Krause–Nikolaus or of Burklund–Hahn–Levy–Schlank was found as of 29 October 2026 (both still arXiv version 1).
8. Not covered: Vandiver's conjecture. Weibel's history of K-theory before 1980 was not found online; his K-book was used instead.
9. Claims resting on a single source: Serre's 1958 method (only his exposé), the modular-representation method of Quillen's 1971 proof (only MacTutor), Bloch's 1974 date and Waldhausen's 1985 date (only Wikipedia's reference list), and the Dundas–Goodwillie–McCarthy input to Nikolaus–Scholze (only a section title in Rognes).
10. The dates 1962 (Bass–Schanuel), 1967 (Bass–Murthy) and 1968 (Bass's negative K-groups), and 1975 for the Quillen–Lichtenbaum conjecture, are given by Wikipedia; the 1964 Bass–Heller–Swan paper is confirmed at Numdam [17]. The step from Merkurjev–Suslin 1982 to the degree-3 case is stated by Wikipedia only; the 1990/1991 date of the degree-3 paper is confirmed at Math-Net.Ru [42].

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Lie groups

Lie groups are groups that are also smooth manifolds, such as the rotations of space; their infinitesimal versions are Lie algebras. The chain starts with Sophus Lie's transformation groups (1869–74) and Wilhelm Killing's Lie algebras, which were found independently. Killing's classification, made rigorous by Élie Cartan in 1894, feeds Weyl's representation theory (1924–26), the Peter–Weyl theorem and Harish-Chandra's discrete series.

Hilbert's fifth problem (1900) was solved case by case: by von Neumann for compact groups, using Haar measure, and by Pontryagin for abelian groups. Gleason and Montgomery–Zippin solved it in full in 1952. Its generalisation, the Hilbert–Smith conjecture, is still open in 2026.

The lattice line runs from Kazhdan's property (T) to Margulis's superrigidity, expanders and the Oppenheim conjecture, and then to Ratner's theorems. Mostow rigidity inspired superrigidity but was not a step in its proof: the one source that links them, Fisher, calls it inspiration. In 2025–26 the front is effective Ratner theory: Lindenstrauss–Mohammadi–Wang (listed by the Annals) and Lindenstrauss–Mohammadi–Yang (a preprint as of 29 October 2026).

Lie, Killing and Cartan: the classification (1869–1932)

From 1869 to 1874 Sophus Lie built the theory of continuous transformation groups [1, 2]. He worked through infinitesimal transformations, his "infinitesimal group", which is now called the Lie algebra. The theory became systematic from the winter of 1873–74, ten years before Killing's work of 1884. It was later set out in three volumes with Friedrich Engel (1888–93). Felix Klein's ideas influenced Lie, but only as influence: the two met in 1869 and 1870 [1].

Around 1880 Wilhelm Killing introduced Lie algebras from an infinitesimal approach to space forms; he published them in his Braunsberg Programmschrift of 1884 [3, 4] (note 3). He found them independently of Lie.

On 12 April 1886 Killing conjectured that the only simple algebras are those of SL and the orthogonal groups [3, 5]. The conjecture was false. On 23 May 1887 he wrote to Engel that he had found a further simple algebra, now called G_2, together with its root system [3, 4, 5]. This refuted his own conjecture. Agricola lists G_2 among the five exceptional algebras, apart from the series sl, so and sp [5]. (A quick check: G_2 has 12 roots, closed under the reflections in all of them, as all 144 pairs confirm; its dimension is 12 + rank 2 = 14.)

By 18 October 1887 Killing had a complete list of the simple complex Lie algebras, including the exceptional ones [3]. He printed the classification in four parts in Math. Ann. (1888–90) [3, 2, 4]. The Cartan subalgebra, the Cartan matrix and root systems were all first introduced here. (A quick check, for n = 1 to 5: type A has dimension n(n + 2), giving 3, 8, 15, 24, 35; types B and C have n(2n + 1), giving 3, 10, 21, 36, 55; type D has n(2n − 1), giving 1, 6, 15, 28, 45. As roots and dimension (roots + rank): E_8 240/248, E_7 126/133, E_6 72/78, F_4 48/52, G_2 12/14.) The classification grew out of

his G_2 discovery; MacTutor places both inside the same work rather than as separate steps.

In 1894, four years after Killing's last part, Élie Cartan completed the classification of the semisimple complex Lie algebras in his thesis [6, 4]. He also constructed the exceptional algebras. MacTutor calls the thesis "essentially a rigorous rewriting of Killing's paper" [6]. It rests on Killing's classification, which it repairs.

In 1913 Cartan found spinors while determining the linear representations of the orthogonal groups [7]; MacTutor says of the simple groups [6] (note 2). From 1926 to 1932 he classified the Riemannian symmetric spaces through the involutive automorphisms of compact Lie algebras (Bull. SMF 54, 1926 and 55, 1927; completed by 1932) [6, 8].

Invariant integration and representations (1897–1966)

In 1897 Adolf Hurwitz introduced an invariant integral on Lie groups [9]. He used it over the rotation and unitary groups to generate invariants [10, pp. 70–90]. It does not rest on an earlier link in this chain.

From 1924 to 1926 Hermann Weyl developed the representation theory of compact semisimple groups and proved his general character formula [11, 2, 10]. His work separates Lie algebras from Lie groups. (A quick check: the Weyl dimension formula $(a + 1)(b + 1)(a + b + 2)/2$, over highest weights (a, b) with $a + b \le 4$, gives SU(3) irreducibles of dimension 1, 3, 6, 8, 10, 15, 24 and 27.) Weyl builds on Cartan's classification, since he treats the exceptional groups too [10]; no source says how he used it. He also rests on the integration of Hurwitz and Schur: Weyl integrated class functions over Cartan subgroups, after Schur 1924 [11, 10].

In 1927 Fritz Peter and Weyl proved the Peter–Weyl theorem (Math. Ann. 97) [12, 13, 10]. The matrix coefficients of the irreducible unitary representations of a compact group, normalised by $\sqrt{\dim}$, form an orthonormal basis of $L^2(G)$. Orthogonality comes from invariant integration. Completeness comes from integral operators, in Hirai's words "from the theory of eigenvalues" [10, §5, pp. 749–752]. (A quick check: the SU(2) characters χ_0 to χ_4, integrated against the Haar density $(2/\pi)\sin^2\theta$ on $[0, \pi]$ by Gauss–Legendre quadrature with 200 nodes, give the 5×5 identity Gram matrix, with largest deviation 6×10^{-15}; for the group S_3, $1^2 + 1^2 + 2^2 = 6$.) The theorem rests on Weyl's work of 1924–26, through orthogonality relations for invariant measures that build on Schur [10] (note 9). The modern form on $L^2(G)$ uses Haar measure, which came six years later; Haar measure is background to that form, not a step in the 1927 proof (note 8).

In 1932 Alfréd Haar found the Haar measure on second-countable locally compact groups; it was printed in the Annals in 1933 [14, 9]. This generalises Hurwitz's integral: Hurwitz's 1897 integral is the special case for Lie groups [9]. Haar's introduction cites Hurwitz, Schur and Weyl, according to Hirai [10]. That is a citation, so the dependency is one of ideas.

In 1940 André Weil and Henri Cartan proved the general existence and uniqueness of Haar measure [9, 15]. Weil's proof uses the axiom of choice; Cartan's does not. This was seven years after Haar. It generalises Haar 1933: Weil's existence proof builds on Haar's ideas, and Cartan's shows that existence does not depend on choice [15].

In 1946 Claude Chevalley published *Theory of Lie Groups*, with further volumes in

1951 and 1955 [16, 2]. It was the first systematic exposition of the foundations, from the global viewpoint of analytic manifolds. It reworks Lie's theory in modern language.

From the 1940s to the 1950s algebraic groups were developed by algebraic means [17, 18]. Ellis Kolchin (1948) worked in any characteristic, "without any recourse to" the Lie algebra (note 9). Armand Borel (Annals 64, 1956) laid the global foundations. Chevalley's classification seminar (1956–58) classified the groups over any algebraically closed field [17, 18]. This line does not rest on an earlier link in this chain.

In 1965–66 Harish-Chandra constructed the discrete series of semisimple groups [19, 20, 21, 22, 23, 24]. He used invariant eigendistributions (Acta 113, 1965) and then gave explicit characters and a classification (Acta 116, 1966). A discrete series exists if and only if rank G = rank K [25]; in Arthur's form, if and only if G has a compact Cartan subgroup [19]. The Plancherel formula followed in 1976 (Annals 104). The work generalises Weyl's character formula: Arthur says his integration formula generalises Weyl's, and MacTutor calls the character formula an analogue of Weyl's [19, 20]. The years are taken from the IAS repository [21, 22] (note 6).

Hilbert's fifth problem (1900–2024)

In 1900, in his address to the International Congress of Mathematicians, David Hilbert posed his fifth problem: characterise Lie groups without assuming differentiability [26, 27]. It took 52 years to solve in full.

In 1933 John von Neumann solved it for compact groups, "introducing analytical parameters", in the same Annals issue as Haar's paper [28, 9]. That was 33 years after Hilbert posed it. It solves one case of Hilbert's problem and rests on Haar's measure.

In 1934 Lev Pontryagin solved the abelian case through characters [29, 26]. This rests on his duality theory for locally compact abelian groups, developed in 1931–34 and published in Annals 35 (1934) [30, 31]; a book followed in 1938. (A quick check: the 6 characters of Z/6 give (1/6)·CC* equal to the identity, with largest deviation 9×10^{-16}, so they are orthonormal; from 1900 to 1934 is 34 years.) It solves another case of Hilbert's problem (note 4).

In 1952 Andrew Gleason, and Deane Montgomery with Leo Zippin, solved Hilbert's fifth problem [32, 26, 33]. A locally Euclidean topological group is a Lie group. The proof goes through the no-small-subgroups criterion. Montgomery and Zippin's book followed in 1955 (note 1).

In 1953 Hidehiko Yamabe proved what is now the Gleason–Yamabe theorem [34, 27, 26]. A connected locally compact group is a projective limit of Lie groups, and having no small subgroups implies being Lie. This removes the assumption of finite dimension. It rests on Gleason and Montgomery–Zippin 1952.

From 2011 to 2013 John Pardon proved the Hilbert–Smith conjecture for 3-manifolds (JAMS 2013) [35, 26]. Every locally compact group acting faithfully on a connected 3-manifold is a Lie group. He proves this by ruling out faithful actions of the p-adic integers Z_p. (A quick check: arXiv December 2011 to JAMS 2013 is two years.) The Hilbert–Smith conjecture generalises the fifth problem to group actions [26], so this work builds on the 1952 solution; no source states that the proof uses the 1952

theorem. The reduction to Z_p actions is a known reduction that Pardon uses, not his own (note 5).

In 2024 Egor Shelukhin proved a symplectic Hilbert–Smith theorem [36]. Z_p has no nontrivial continuous action by Hamiltonian homeomorphisms on symplectically aspherical manifolds. The proof uses barcodes and power operations in Floer cohomology. It appeared on arXiv as 2403.07195 (version 2, June 2024), with no journal recorded as of 28 October 2026. It uses the reduction of Hilbert–Smith to Z_p actions [26, 35], though Shelukhin's abstract does so only implicitly.

Lattices, rigidity and homogeneous dynamics (1962–2026)

In 1962 Borel and Harish-Chandra proved that arithmetic subgroups of semisimple groups have finite covolume, so they are lattices (Annals 75) [37, 38, 39]. Morris gives the result as Theorem 1.24 of his *Introduction to Arithmetic Groups* [39]. It does not rest on an earlier link in this chain.

In 1967 David Kazhdan introduced property (T) (Funkts. Anal. 1:1) [40, 41]. The trivial representation is isolated in the unitary dual. Consequences include the finite generation of lattices. Property (T) is 59 years old in 2026.

In 1967–68 George Mostow proved his rigidity theorem (Publ. IHÉS 34; received 10 April 1967, printed in 1968, a year later) [42, 43]. Closed hyperbolic manifolds of dimension 3 or more are determined by their fundamental group. As the paper's title says, the proof goes through quasi-conformal mappings. Gopal Prasad (1973) and Albert Marden (1974) extended it to finite volume [43] (note 9).

In 1973 Grigory Margulis constructed the first explicit families of bounded-degree expander graphs (Probl. Peredachi Inf. 9(4)) [44, 45, 46]. The construction comes from property (T) of SL(n, Z) for $n \geq 3$ [41], so it rests on Kazhdan's property (T). (A quick check: ten random 8-regular graphs on 200 vertices, with seeds 1973 to 1982, have second eigenvalue between 4.94 and 5.24, well below the degree 8; the Ramanujan bound is $2\sqrt{7} \approx 5.29$.)

In 1974, six years after Mostow, Margulis proved superrigidity for lattices in higher-rank semisimple groups [44, 47, 48, 49]. This gives their arithmeticity and proves the Selberg–Piatetski-Shapiro conjecture. Fisher dates it to 1974 (ICM Vancouver); the LMS citation says "the 1970s" [48, 49]. Fisher writes that the proof was "inspired by Mostow's study of boundary maps in his rigidity theorem" [48]. That is influence, not a proof step, so Mostow rigidity and superrigidity count as separate results here. No source consulted says that superrigidity rests on Borel–Harish-Chandra (note 10). Margulis received the Fields Medal in 1978 [47].

In 1986–87 Margulis proved the Oppenheim conjecture [44, 50, 49, 51, 52]. Oppenheim had posed it in 1929, then for five or more variables [51]. An indefinite irrational quadratic form Q in 3 or more variables takes values $|Q(x)| < \varepsilon$ at non-zero integer vectors, for every $\varepsilon > 0$ [51, 52]. Its values are then dense [51] (note 9). Madabusi Raghunathan had reduced the problem to unipotent flows, and Margulis proved the case needed [44, 49]. The proof thus rests on Raghunathan's reduction. MacTutor dates it 1986; Wikipedia and the Abel Prize booklet say 1987 (note 7).

In 1990–91 Marina Ratner classified the invariant measures, orbit closures and equidistribution of unipotent flows [50, 53]. This proved Raghunathan's measure and topological conjectures, posed around 1980, 11 years earlier. The papers appeared in

Invent. 101 and Acta 165 (1990), and Annals 134 and Duke 63 (1991); only Wikipedia's references give these details (note 9). (A quick check: from Margulis in 1986–87 to Ratner in 1991 is 4 to 5 years.) The work generalises Margulis's Oppenheim proof: MacTutor says Margulis had proved a special case of Raghunathan's conjecture, and Wikipedia says unipotent flows were decisive in his proof [50, 53]. Ratner received the Ostrowski Prize in 1994 [50] (note 7).

From 2017 to 2020 Aaron Brown, David Fisher and Sebastian Hurtado proved Zimmer's conjecture for C^2 actions of finite-index subgroups of SL(m, Z), m > 3 [54, 55, 56, 48]. It was published in Invent. Math. 221 (2020), 1001–1060 (arXiv 1710.02735, version 3 of March 2020). The proof uses results of Lubotzky–Mozes–Raghunathan, of Kleinbock–Margulis and de la Salle's strong property (T), as the paper's abstract states [54]. A link from superrigidity is not documented (note 11).

In 2020 Hillel Furstenberg and Margulis received the Abel Prize "for pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics" [57, 44].

From 2022 to 2025 Elon Lindenstrauss, Amir Mohammadi and Zhiren Wang proved effective equidistribution, with a polynomial rate, for unipotent orbits of $SL_2(R)$ in arithmetic quotients of $SL_2(C)$ and of $SL_2(R) \times SL_2(R)$ [58, 59]. The proof uses a Margulis function, incidence geometry and a spectral gap. The paper is listed by the Annals, with no volume shown on 28 October 2026; arXiv version 2 appeared in September 2025. (A quick check: arXiv November 2022 to version 2 in September 2025 is 34 months.) It makes Ratner's theorems effective; this link rests on the authors' own text (note 11).

In 2026 Lindenstrauss, Mohammadi and Lei Yang announced polynomially effective equidistribution for unipotent orbits in S-arithmetic quotients of K-forms of SL_{2^n}, a quantitative form of the theorems of Ratner and Shah [60, 61]. The method uses an adapted projection theorem of Bourgain type. According to the authors' own 10-page research announcement, which outlines the proof (arXiv 2601.09983, 15 January 2026), the result holds; there is no journal publication as of 29 October 2026 [60]. Mohammadi presented the work at the joint IAS/Princeton Groups and Dynamics Seminar on 3 February 2026 [61, 62]. (A quick check: Ratner 1991 to 2026 is 35 years.) It makes Ratner and Shah effective, a link that rests on the authors' own texts (note 12).

Notes

1. One summary said Gleason–Montgomery–Zippin answered Hilbert's question "negatively", but Wikipedia, MacTutor and nLab all state the positive result [26, 32, 27]. MacTutor credits Montgomery alone, under finite dimension, while Iwasawa's review, MacTutor's Gleason page and nLab credit Gleason together with Montgomery–Zippin [34, 32, 27]. The result is dated 1952 by MacTutor and by Wieting (Reed), and "the 1950s" by Wikipedia [32, 33, 26].
2. Spinors are linked to the orthogonal groups by Wikipedia and to the simple groups by MacTutor [7, 6].
3. Killing's Lie algebras are dated c. 1880 by Wikipedia and 1884 by MacTutor: likely the idea around 1880, published in 1884 [4, 3]. G_2 is dated 23 May 1887 by MacTutor and March 1887 (the root system) by Agricola; both put it in spring 1887 [3, 5].

4. Pontryagin's duality (1931–34 in Wikipedia; Annals 35, 1934, in the Encyclopedia of Mathematics) and his abelian case of the fifth problem (1934) are two events, often merged [30, 31, 29]. The earlier "by 1932" date in MacTutor refers to his homology duality, not to groups.
5. Pardon's paper was submitted to arXiv on 11 December 2011 (version 3 of 22 January 2013) and printed in JAMS in 2013 [35].
6. Harish-Chandra's discrete-series papers are Acta 113 (1965) and Acta 116 (1966); the YMSC archive's "1965" for volume 116 is a slip. The Plancherel formula is 1976.
7. Ratner's Ostrowski Prize is dated 8 July 1994 by MacTutor [50]; some accounts give 1993. Margulis's Oppenheim proof is dated 1986 by MacTutor and 1987 by Wikipedia and the Abel Prize booklet [44, 51, 52]. A claim that Margulis arithmeticity holds for SO(n, 1) is very likely garbled and is not used. Weyl's representation theory is dated 1924–26 (MacTutor), not 1925–26. Superrigidity is dated 1974 by Fisher and "the 1970s" by the LMS citation; a date of 1975 rested only on the date of a reference and is not used.
8. Haar measure (1933) came after the Peter–Weyl theorem (1927), so it cannot be a step in that proof; it is background to the modern $L^2(G)$ form only.
9. Claims resting on a single source: the Peter–Weyl completeness method and its link to Weyl (only Hirai) [10]; Kolchin 1948 (only the KSDA paper) [18]; Prasad 1973 and Marden 1974 (only Wikipedia) [43]; the density of values in the Oppenheim conjecture (only Wikipedia's introduction) [51]; the journal details of Ratner's papers (only Wikipedia's references) [53].
10. Fisher's survey does not mention Borel–Harish-Chandra in its account of the arithmeticity proof; it credits Borel only with an earlier converse result [48].
11. The Brown–Fisher–Hurtado link to Fisher's survey is by a co-author [48]. The link from Ratner to Lindenstrauss–Mohammadi–Wang rests on the authors' own texts; the Annals page still showed no volume on 29 October 2026 [59].
12. The Lindenstrauss–Mohammadi–Yang result rests on the authors' own announcement and a co-author's seminar talk; there is no independent source [60, 61, 62]. Not covered, for lack of sources: Cartan's closed-subgroup theorem (1930), van Kampen, the Langlands classification (1973), Breuillard–Green–Tao and Kaluba–Nowak–Ozawa.

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Complex analysis

Complex analysis studies functions of a complex variable that can be differentiated, the holomorphic functions, and the surfaces on which they live. The chain starts with Cardano's and Bombelli's arithmetic of $\sqrt{-1}$ and with Euler's formula. It then runs along two proven lines. The first is Cauchy's integral theorem, repaired by Goursat. The second is Riemann's mapping theorem of 1851, broken by Weierstrass's counterexample and repaired by Hilbert and Osgood, which leads on to uniformization in 1907.

In the 20th century the Bieberbach conjecture (1916) runs through Löwner's equation and the Askey–Gasper inequality to de Branges's proof in 1985. Löwner's equation returns in Schramm's SLE (1999). In 2026 the newest link is Tao's preprint of December 2025. It proves the Erdős–Herzog–Piranian lemniscate conjecture only for all sufficiently large degree; it is not yet published, and the conjecture for all n remains open. The three discoveries of the complex plane were independent of one another, and Gauss's 1811 letter did not lead to Cauchy's work.

Complex numbers and the complex plane (1545–1831)

Around 1545 Gerolamo Cardano gave rules for arithmetic with $\sqrt{-1}$, and in 1572 Rafael Bombelli developed them [1, 2]. Bombelli showed that the casus irreducibilis of the cubic formula, where the formula passes through square roots of negative numbers, still gives the correct real roots. (A quick check: $(2 + i)^3 = 2 + 11i$, and $x = 4$ satisfies $x^3 - 15x - 4 = 0$.)

In 1748 Leonhard Euler published e^(ix) = cos x + i sin x in his *Introductio* [3]. The Encyclopedia of Mathematics cites an earlier paper, in Miscellanea Berolinensia 7 (1743) [4] (note 2). The proof compares the power series of exp, cos and sin [3] (note 9). Roger Cotes already had the equivalent statement ln(cos x + i sin x) = ix [5], dated 1714 by Wikipedia [3]. (A quick check: |e^i − (cos 1 + i sin 1)| = 0.0 in double precision.) Euler's formula uses complex arithmetic of the kind Bombelli developed, but no source states a direct debt (note 10).

The complex numbers were then placed in the plane, three times and independently [6, 1]. Caspar Wessel presented his paper on 10 March 1797; it was printed in 1799. Jean-Robert Argand followed in 1806 and Carl Friedrich Gauss in 1831. In all three, addition is vector addition, and multiplication multiplies lengths and adds angles. None rests on another.

Cauchy's integral theorem (1811–1900)

In 1811 Gauss stated, in a letter to Bessel, that the integral $\int f\,dz$ is independent of the path [7, 8]. He never published a proof. The letter was private, and no source says that Cauchy knew of it; Cauchy's work is independent of it.

In 1814 Augustin-Louis Cauchy wrote a memoir on definite integrals [7, 9, 10]. According to one historian it was not printed until 1827; MacTutor says it was

published in 1814 and "later became the basis of his theory of complex functions" [7, 9] (note 3). In it, the Cauchy–Riemann relations make the integrals over two sides of a rectangle equal. The proof interchanges the order of a double integral [7, pp. 336–340].

In 1825 Cauchy proved his integral theorem in the general case: the integral is independent of the path [7, 8]. He varied the path continuously and assumed that f' is continuous. (A quick check with the 1000-point trapezoid rule on the unit circle: $\oint z^2\,dz$ is of order 10^{-16}, effectively 0; $\oint dz/z = 6.283185i = 2\pi i$; and the integral formula returns $\exp(0) = 1.000000$.) This builds on the 1814 memoir, which MacTutor calls the basis of his theory [9]; the historian's account does not say that the 1825 proof uses it.

In 1826 Cauchy began the calculus of residues [9, 11, 12]. In 1829, in his *Leçons sur le calcul différentiel*, he defined for the first time a complex function of a complex variable [9]. Neither rests on an earlier link in this chain.

In 1884 Édouard Goursat proved Cauchy's theorem again (Acta 4), still assuming f' continuous [7, 8, 13, 14]. In 1900 (Trans. AMS 1, 14–16), at William Osgood's request, he removed that assumption, so the theorem holds for f merely holomorphic. The Encyclopedia of Mathematics calls Goursat's the first complete proof [8]. (A quick check: $1900 - 1884 = 16$ years.) This repairs the f' assumption in Cauchy's 1825 proof (note 1).

Riemann's mapping theorem and uniformization (1851–1923)

In 1851 Bernhard Riemann's dissertation defined holomorphy through the Cauchy–Riemann equations [15, 7, 16]. It gave a direct proof of Green's formula, introduced Riemann surfaces and stated the mapping theorem. Riemann took the map from a minimiser of the Dirichlet integral. The dissertation draws on Cauchy's 1825 work as influence, re-deriving the Cauchy–Riemann equations. It also rests on the Dirichlet principle (see the chapter on potential theory). One historian credits Riemann with the first published proof of Green's formula, while MacTutor credits Green's Essay of 1828 (note 6).

On 14 July 1870, in a lecture to the Berlin Academy, Karl Weierstrass showed that a minimiser of a Dirichlet-type integral need not exist [17, 18]. Admissible functions can drive the integral down to 0, while none attains 0. (For the computation, see the chapter on potential theory.) This is a counterexample to an assumption in Riemann's 1851 argument (note 4).

In 1879 Émile Picard proved the little Picard theorem: a non-constant entire function omits at most one value [19, 20]. The proof composes the function with the modular λ-function and applies Liouville's theorem. It rests on Hermite's modular function.

In 1900, 30 years after Weierstrass, David Hilbert justified the Dirichlet principle (Jahresber. DMV 8, 1900; Math. Ann. 59, 1904) [21, 22]. He used a minimising sequence and made it converge uniformly. This repairs the gap that Weierstrass had found (note 5).

Also in 1900 Osgood gave the first rigorous proof of the Riemann mapping theorem [16, 23, 24, 25]. He used Green's function for an arbitrary simply connected plane

region. The proof repairs Riemann's 1851 argument. It uses Poincaré's methods [23, 24] and Green's function, as the title of Osgood's 1900 paper shows [16, 24].

In 1907 Henri Poincaré and Paul Koebe both proved the uniformization theorem [26, 27, 28]. Every simply connected Riemann surface is conformally the disc, the plane or the sphere. The proof constructs a harmonic function, often a Green's function [26]. In Weyl's words, quoted by MacTutor, both combined "H A Schwarz's idea of the universal covering surface" with "simple estimates of the Harnack type for harmonic functions" [27]. The theorem answers Hilbert's 22nd problem [27, 28]. It follows the attempts of Klein and Poincaré, in 1882–83, to prove it by the "continuity method", which MacTutor dates "more than twenty years before" [26, 27]. Uniformization generalises the mapping theorem [26], but no source says that the 1907 proofs use Osgood's, so the two are separate results here.

The mapping theorem then received new proofs by Paul Montel's normal families [16, 29, 25, 30]. Constantin Carathéodory gave one in 1912, by an iterative method whose convergence follows from Montel's theorem [30]; full details appeared in the Schwarz Festschrift of 1914 [30]. Koebe gave elementary iterative proofs between 1907 and 1915 [30]. In 1922 Lipót Fejér and Frigyes Riesz obtained the map from an extremal problem: they maximise $|f'(z_0)|$ over univalent maps into the disc, again with normal families [25, 30]. Their proof was published by Tibor Radó in 1922–23 [30]. Carathéodory wrote in 1928 that it "was reserved for the Hungarian mathematicians L. Fejér and F. Riesz" to choose a variational problem whose solution is known to exist, unlike Dirichlet's principle [30]. Like Riemann's own approach, this is an extremal problem [16]. Carathéodory proved the extension of the map to the boundary in 1913 [16]. All these proofs rest on Montel's normal families (note 7).

The Bieberbach conjecture and its offshoots (1916–2025)

In 1916 Ludwig Bieberbach proved $|a_2| \le 2$ for schlicht (univalent) functions $f(z) = z + a_2z^2 + \dots$ on the unit disc [31, 32]. The method is not described in the sources consulted. In a footnote he conjectured that $|a_n| \le n$ for all n. (A quick check: the series of the Koebe function $z/(1 - z)^2$, to order 7, has coefficients 1, 2, 3, …, 7, so $a_n = n$ and the function is extremal.)

In 1923 Karl Löwner (later Charles Loewner) proved $|a_3| \le 3$ [31, 32, 33]. He introduced the parametric method of the Löwner differential equation. The work rests on Bieberbach's problem of 1916.

The next cases came much later [32, 33]. In 1955 Paul Garabedian and Max Schiffer proved the case $n = 4$, combining variational and parametric methods. In 1972 Roger Pederson and Schiffer proved the case $n = 5$ by variational methods. Both rest on Bieberbach's problem.

In 1976 Richard Askey and George Gasper proved that $\Sigma_{k\le n} P_k^{(\alpha,0)}(x) \ge 0$ for Jacobi polynomials, a positivity result for hypergeometric functions [32, 34, 33] (note 9). The result belongs to the theory of special functions (MSC 33) and does not rest on an earlier link in this chain. (A quick check, on 4001 equally spaced points x in $[-1, 1]$, with $n \le 20$ and $\alpha = 0, 1, 2, 4, 6$: the minimum is -7×10^{-16}, a rounding of 0 at $x = -1$, never truly negative.)

In 1977 Isaak Milin's logarithmic-coefficient conjecture became available in English; it had appeared in his Russian book of 1971 [32, 34] (note 8). It implies Robertson's

conjecture of 1936, and that in turn implies Bieberbach's. Milin also gave, with Lebedev, the exponentiation inequalities that make this work. These do not rest on an earlier link in this chain.

In 1984 Louis de Branges proved the Bieberbach conjecture; the proof was printed in Acta Math. in 1985, 69 years after Bieberbach [34, 32, 33]. He proved Milin's conjecture. He used a Löwner flow with a monotone functional, whose positivity reduces to the Askey–Gasper inequality. The proof rests on Löwner 1923, on Askey–Gasper 1976 and on Milin 1977, and it settles Bieberbach's conjecture of 1916.

In 1999 Oded Schramm introduced the stochastic Löwner evolution, SLE; the journal version appeared in 2000 [35, 36]. It is the Löwner equation driven by Brownian motion. Schramm proposed it as the scaling limit of loop-erased random walk and of the uniform spanning tree, and suggested a variant for critical percolation [35]. Conformal invariance and a Markov property allow the random curve to be encoded by a Brownian driving function [36]. The work belongs to probability (MSC 60) and rests on Löwner 1923.

In 1958 Paul Erdős, Fritz Herzog and George Piranian asked whether the lemniscate $\{|p| = 1\}$ of a monic polynomial p of degree n is longest for $p = z^n - 1$ [37, 38, 39]. This is the Erdős–Herzog–Piranian conjecture.

In 2009 Alexander Fryntov and Fedor Nazarov proved that $z^n - 1$ is a local maximiser; their paper had appeared on arXiv on 5 August 2008 [38, 37, 40, 41]. They also proved that the maximal length f(n) satisfies f(n) ≤ 2n + O(n^(7/8)) [38; 37, Table 1, from their §9]. Their abstract states 2n + o(n) [40]. This does not rest on an earlier link in this chain.

In December 2025 Terence Tao proved the Erdős–Herzog–Piranian conjecture for all sufficiently large n, with $z^n - 1$ as the unique maximiser [42, 38, 40]. It is a preprint (arXiv version 1 of 13 December 2025, version 2 of 22 December 2025), with no journal reference. The full conjecture remains open. (A quick check: with L = ∫ over [−π, π] of (2|cos(θ/2)|)^(−(n−1)/n) dθ, the length of the lemniscate of $z^n - 1$ minus 2n is 4.283 for n = 1, 3.180 for n = 3, 2.886 for n = 10 and 2.778 for n = 200, so it stays bounded; and 2025 − 1958 = 67 years.) The proof settles the 1958 conjecture for large n only. It builds on Fryntov–Nazarov 2009, on their Stokes-theorem identities and their §9 bound [42]. Tao gives no explicit threshold for the degree: his constants are "effectively computable" but not optimised. Erdős Problems, last edited on 23 January 2026, still lists the problem as open [38].

Notes

1. The Dictionary of Scientific Biography entry on MacTutor credits the removal of the continuity of f′ to one of Goursat's earliest works and calls the 1900 paper essentially the same material [14]. The historian's account says that the 1884 proof still assumed it and that the 1900 paper, written at Osgood's request, removed it [7]. The text follows the latter. That account attributes the method of subdividing a triangle to Briot and Bouquet [7].
2. Euler's formula is dated 1748 (*Introductio*; Wikipedia) and 1743 (Miscellanea Berolinensia 7; Encyclopedia of Mathematics) [3, 4].
3. Cauchy's 1814 memoir was "not published until 1827" according to arXiv 2311.16649, while MacTutor says he published it in 1814 [7, 9]. Cauchy's

theorem is variously dated 1814, 1825 or 1831; the 1831 date is not supported by any source consulted.

4. Weierstrass's critique is dated 1870 by Wikipedia, Archibald and Pokroky, "late 1860s" by the Encyclopedia of Mathematics, and "after 1857" by MacTutor [17, 18, 21, 22, 15]. The dated lecture is 14 July 1870.
5. Hilbert's repair is given as 1900, as 1901 (MacTutor's Riemann page) and as 1905 (a Wikipedia footnote); Pokroky gives 1900 and 1904 [21, 15].
6. Green's formula: arXiv 2311.16649 credits Riemann (1851) with the first published proof, MacTutor credits Green's Essay of 1828 [7, 15]. This needs a check against the primary sources.
7. The Fejér–Riesz proof is dated 1922 by Wikipedia and 1923 by the Alexander paper [16, 25]; Gabard's survey reconciles the two: found in 1922 and published by Radó in 1922–23 [30]. Wikipedia attributes a normal-families proof to Koebe in 1914; Gabard lists Koebe's proofs of 1907, 1909, 1912 and 1915 and no 1914 paper [16, 30].
8. SLE is dated 1999 (arXiv) or 2000 (journal; Wikipedia) [35, 36]. De Branges's proof is dated 1984 (MacTutor's Steele citation; Encyclopedia of Mathematics) or 1985 (print) [34, 33]. Milin's conjecture is in his Russian book of 1971, translated in 1977 (Wikipedia's references) [32]. Fryntov–Nazarov was on arXiv on 5 August 2008; 2009 is the year Erdős Problems gives [40, 38].
9. Claims resting on a single source: Euler's power-series method (only Wikipedia) [3] and the year 1976 for Askey–Gasper (only Wikipedia) [32].
10. A debt of Euler to Bombelli is not stated in any source consulted; MacTutor's Bombelli page does not mention Euler [2].
11. Not covered: the subclass lists of msc2020.org and zbMATH were not checked. Cauchy's integral formula (1831) and Morera's theorem (1886) were searched for, but two sources were not found. Weierstrass factorisation, Mittag-Leffler, Hadamard, Nevanlinna theory and quasiconformal maps were not researched, so the chain is thin in 1876–1925, and in 1925–1985 outside the Bieberbach line. The status of the Riemann mapping theorem and of de Branges's theorem in Lean's Mathlib was not checked.

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Potential theory

Potential theory began in celestial mechanics. Newton's law of attraction led to Lagrange's potential function, to Laplace's equation and to Poisson's correction inside matter. Green (1828) and Gauss (1840) then supplied the integral identities and the very word "potential". The proof line proper runs through the Dirichlet problem: Riemann assumed the Dirichlet principle, Weierstrass refuted it as stated (1870), Schwarz and Neumann went round it, and Hilbert repaired it (1900).

The irregular boundary points of Zaremba and Lebesgue then led to Perron's method (1923) and Wiener's capacity criterion (1924), and Kakutani (1944) tied the whole problem to Brownian motion. The modern line is geometric. It runs from Makarov's dimension theorem (1985) through the rectifiability of harmonic measure (2016) to the proof of Carleson's ε^2-conjecture (2021), and on to the results of the European GAPT project up to November 2025. Twenty-three links lead from Newton to that project. Makarov's theorem stands apart: no source states that the later rectifiability results rest on it.

From gravitation to the potential, 1687–1840

In 1687 Newton stated his law of gravitation, and the forces that obey it define the field that potential theory studies [1, 2]. In 1773 Lagrange showed that gravitational fields are potential fields: the force is the gradient of a single scalar function, the potential function [1, 3]. The problem he answered was the one Newton's law had posed; the Encyclopedia of Mathematics states this dependency [1].

Laplace then showed that the potential V satisfies $\Delta V = 0$ outside the attracting masses. The equation appears in his work of 1782–1799 and in the *Mécanique Céleste*, alongside Legendre's work of 1784–94 [1, 4]. Laplace expanded the potential in the Legendre functions that were long called Laplace coefficients [4]. The date of the equation varies between sources (note 1).

In 1813 Poisson corrected Laplace's equation inside matter. There $\Delta V = -4\pi\rho$, where ρ is the density of mass at the point [5, 6]. This repairs Laplace's equation where it fails. (A check: for a uniform ball with $V = \int \rho/|x - y|$ and $G = 1$, the potential is $2\pi\rho(R^2 - r^2/3)$ inside and $(4/3)\pi\rho R^3/r$ outside. Sympy gives $\Delta V = -4\pi\rho$ inside and 0 outside, and both V and its derivative are continuous at $r = R$.) Wikipedia dates the equation to 1823 (note 2).

In 1828 George Green published his *Essay*. It named the "potential function", gave Green's theorem relating surface and volume integrals, and introduced Green's function for boundary problems [7, 1]. The *Essay* stood on no earlier link in this chain. It spread through William Thomson, who found it in 1845, seventeen years after it appeared ($1845 - 1828 = 17$) [7, 8]. According to Wikipedia, Riemann and Carl Neumann later gave "Green's function" its name (note 3).

In 1840 Gauss shortened Green's "potential function" to "potential" [1, 9]. Who first introduced the potential function is itself disputed between Lagrange, Green and Laplace (note 4).

The Dirichlet principle and its repair, 1851–1904

Before 1851 Dirichlet taught in his Berlin lectures what Riemann later called the Dirichlet principle. The harmonic function with given boundary values is the one that minimises the Dirichlet integral, assuming a minimum exists [10, 12]. Gauss, Green and Thomson had already used the idea, and Riemann relied on it in 1851 (see the chapter on complex analysis) [10, 11]. (A check of the mean-value property behind it: the mean of the harmonic function $u = x^2 - y^2 + 3x$ over 1000 equally spaced points on the circle of radius 0.5 about (0.3, 0.2) is 0.95, which equals u at the centre.)

In a lecture of 14 July 1870 Weierstrass showed that the infimum need not be attained [12, 13]. Functions with an ever steeper jump drive the integral towards 0, yet no admissible function reaches 0. This is a counterexample to the Dirichlet principle as stated (note 5). (A check: for $J(u) = \int x^2 u'^2\,dx$ over $[-1, 1]$ and $u_\varepsilon = \arctan(x/\varepsilon)/\arctan(1/\varepsilon)$, numerical integration gives 0.0634, 0.00637, 0.000637 and 6.37×10^{-5} for $\varepsilon = 0.1, 0.01, 0.001$ and 0.0001. So J tends to 0, close to $2\varepsilon/\pi$, while the limit of u_ε is a step function.)

In 1869–70 H. A. Schwarz, and from 1870 Carl Neumann, proved existence for the Dirichlet problem without the principle [10, 13, 14]. Their method alternates between overlapping subdomains whose Dirichlet problems are already solved; in Schwarz's case these were a circle and a square [14, 15]. Neumann's method was analogous [15]. These results did not rest on Weierstrass's critique: Schwarz's work of 1869 predates the lecture of July 1870, so the two were parallel responses to the same weakness (note 6).

In 1900 Hilbert justified the Dirichlet principle after all [16, 17]. He took a minimising sequence and made it converge uniformly. The work appeared in the *Jahresbericht der DMV* 8 (1900) and in *Mathematische Annalen* 59 (1904). It repairs the gap Weierstrass had exposed thirty years earlier ($1900 - 1870 = 30$). Other dates are also given (note 7).

Irregular points, capacity and Brownian motion, 1910–1958

In 1910 Zaremba showed that the Dirichlet problem cannot be solved on a punctured disc [1, 18]. A harmonic function cannot take an arbitrary value at an isolated boundary point. The example itself appears without attribution in a modern text [19]. The year is 1910 or 1911 (note 8).

In 1912 Lebesgue gave an irregular boundary point in space, for dimension $n \geq 3$ [18, 1]. It is the tip of a sufficiently sharp spike pointing into the region, the Lebesgue spine, or the vertex of a sufficiently acute angle [18, 20]. His papers on it appeared in 1913 and 1924 [18].

In 1923 Perron introduced his method in *Mathematische Zeitschrift* 18 [21, 22]. The solution is the upper envelope of all subharmonic functions that lie below the boundary data. Perron's method builds on the problem that Zaremba and Lebesgue had posed. The Encyclopedia of Mathematics lists their examples before Perron, and the dependency follows from that order [21].

In 1924 Wiener gave his criterion. A boundary point x_0 is regular exactly when $\Sigma\ 2^{k(n-2)}\ \mathrm{cap}(E_k) = \infty$, where E_k is the part of the complement in the k-th dyadic

shell around x_0 [21, 18]. He measured the complement with capacity, shell by shell. Wiener also gave a generalised solution for any continuous boundary data [21, 22]. This rests on Perron's 1923 method [21].

In 1933 Evans proved Kellogg's conjecture: the set of irregular boundary points has outer capacity zero [1, 23]. The result appeared in *Proceedings of the National Academy of Sciences* 19 (1933) [23, 24]. The Encyclopedia of Mathematics states that it rests on Wiener's 1924 criterion [1].

In 1938–40 Brelot showed that a boundary function is resolutive exactly when it is integrable with respect to harmonic measure [21, 1]. This is the Perron–Wiener–Brelot method. It extends Perron's families to arbitrary boundary functions and rests on Perron 1923 and Wiener 1924 [21, 25]. The date is given as 1939 or 1938–40 (note 9).

In 1944 Kakutani showed that Brownian motion solves the Dirichlet problem: $u(x) = E[f(B_\tau)]$, where τ is the time the path first leaves the domain [26, 27]. Brownian exit distributions are harmonic measure, but complete proofs were missing until Doob in 1954 [25]. This was a new method for the old Dirichlet problem. (A check: a walk-on-spheres simulation from (0.3, 0.2) in the unit disc, with 20 000 paths, seed 1944, stopped within 10^{-6} of the circle and boundary data $u = x^2 - y^2 + 3x$, gives 0.963 ± 0.032 at two standard errors. The exact value is 0.95 (note 10).)

In 1950 Choquet introduced capacity and capacitable sets [28]. His 165-page "Theory of capacities" of 1953–54 proved the capacitability theorem and introduced the integral with respect to a monotone, non-additive measure, now called the Choquet integral [29]. It stood on no stated earlier link (note 11).

In 1957 Doob, and in 1957–58 Hunt, built probabilistic potential theory: Markov processes and their potentials, and conditional Brownian motion [30]. Doob built on Kakutani's 1944 work and in 1954 supplied the proof that Kakutani's had lacked [27, 25]. Hunt's theory contains Doob's results as special cases [25].

The geometry of harmonic measure, 1985–2025

In 1985 Makarov proved that harmonic measure on a simply connected planar domain has Hausdorff dimension 1 [31, 32, 33]. The result appeared in *Proceedings of the London Mathematical Society* 51. The sources consulted do not describe his method. No source states that the later rectifiability results rest on this theorem, so they count here as independent of it (note 12). (Thirty-six years separate it from the 2021 result below: $2021 - 1985 = 36$.)

In 2015–16 Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa and Volberg proved that if harmonic measure ω is absolutely continuous with respect to n-dimensional Hausdorff measure on a set E in the boundary, then ω restricted to E is n-rectifiable [34, 35]. The proof goes through the boundedness of the Riesz transform. It uses the rectifiability theorem of Nazarov, Tolsa and Volberg, as the paper's introduction states (note 13). The paper appeared in *GAFA* 26.

In 2021 Jaye, Tolsa and Villa proved Carleson's ε^2-conjecture [36, 37, 38]. Up to a set of zero length, the tangent points of a Jordan curve are exactly the points where the ε^2-square function is finite. The proof appeared in *Annals of Mathematics* 194 (1), 97–161. It settles Carleson's conjecture, and the result sits beside the Bishop–Jones tangent theorem (note 14).

The European ERC project GAPT reported further results for June 2024 to November 2025 [39]. One is a higher-dimensional ε^2-theorem by Fleschler, Tolsa and Villa, in *Inventiones Mathematicae* 241 (2025), 207–307 [40, 41]. It extends the 2021 theorem to higher dimensions [38, 41]. Another is a dimension drop for harmonic measure: for certain flat sets of fractional dimension lying in a hyperplane, harmonic measure has dimension strictly smaller than the set itself, as Tolsa showed [39, 42]. A third concerns the L^p regularity problem in rough domains. Mourgoglou and Tolsa showed that for domains with uniformly rectifiable boundaries the Dirichlet problem with L^p′ data is solvable exactly when the regularity problem is [39, 43].

Notes

1. Laplace's equation is dated 1782–1799 by the Encyclopedia of Mathematics, 1786 by Wikipedia and 1799 by MacTutor, which places it in the *Mécanique Céleste* and says it was known before Laplace. The question is not settled.
2. Poisson's equation is dated 1813 by MacTutor and Rouse Ball, against 1823 in Wikipedia. Two scholarly sources give 1813.
3. Only Wikipedia states that Riemann and Carl Neumann named Green's function; a further wiki credits Riemann alone.
4. The potential function is credited to Lagrange (1773: Encyclopedia of Mathematics, Durand), to Green (MacTutor: he "coined the term 'potential' ") and to Laplace (MacTutor's Laplace biography). The Encyclopedia of Mathematics has Green's "potential function" shortened to "potential" by Gauss in 1840.
5. The Encyclopedia of Mathematics places Weierstrass's critique in the "late 1860s"; the dated lecture is 14 July 1870.
6. Schwarz's alternating method is dated 1869 (Encyclopedia of Mathematics) or 1869–70 (Wikipedia, MacTutor). That is before Weierstrass's lecture of July 1870, so a dependency on the critique would run backwards in time. The claim that Schwarz worked "at Weierstrass's request" is in none of the sources.
7. Hilbert's repair is dated 1900, 1901 and 1905 in different places; Pokroky gives 1900 and 1904.
8. Zaremba's example is 1910 on the Encyclopedia of Mathematics potential-theory page and 1911 on its Perron-method page. Lebesgue's spine is 1912 (Encyclopedia of Mathematics, Wikipedia) and 1913 (the Perron-method page), with papers in 1913 and 1924.
9. Brelot's resolutivity theorem is 1939 on the Encyclopedia of Mathematics potential-theory page and 1938–40 on its Perron-method page.
10. Monte Carlo estimates depend on the implementation; an earlier run with the same seed gave 0.9609 ± 0.0323, and an unseeded run 0.9629 ± 0.0161. All agree with the exact value 0.95.
11. Choquet's capacities are dated 1950 (Wikipedia) and 1953–54 (MacTutor): the first is the notion, the second the 165-page paper.
12. Frostman's work of 1935 and H. Cartan's fine topology belong to this field but are not traced in this chapter.
13. Only the 2016 paper itself states its use of the Nazarov–Tolsa–Volberg theorem.
14. Tolsa's 2025 survey states the Bishop–Jones tangent theorem as its Theorem 2.2 but does not say the 2021 proof uses it. Nor do the survey and the CRM note say that the higher-dimensional proof uses the 2021 one; the extension is stated, the dependency inferred.

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Several complex variables

The theory of several complex variables begins where the methods of one variable stop working. Its first links are Cousin's problems (1895), Hartogs's extension theorem (1906) and Poincaré's proof that the ball and the bidisc differ (1907). The central line is the Levi problem, posed in 1911 and solved by Oka for n = 2 (1942) and in all dimensions by Oka, Bremermann and Norguet (1953–54). Meanwhile Oka's coherence theorem (1950) became sheaf theory in Cartan's Theorems A and B and in Serre's GAGA (1956).

Three later lines follow. The ∂̄-Neumann line runs from Kohn (1963–64) and Hörmander (1965) to Fefferman (1974) and beyond. The Oka line runs from Gromov (1989) to the theorem, published in 2026, that every projective Oka manifold is elliptic. The hyperbolicity line runs from Kobayashi's conjecture (1970) to Brotbek (2016–17) and Bérczi–Kirwan (2023). In all there are 29 links, from Cousin in 1895 to Forstnerič and Lárusson in 2026. Two expected dependencies are not stated by any source: of the Levi solutions on Cartan–Thullen, and of Fefferman on Kohn.

Beyond one variable, 1895–1934

In 1895 Pierre Cousin solved what are now called the Cousin problems, the prescription of zeros and of poles, for certain simple domains, in *Acta Mathematica* 19 [1, 2]. These were products of plane domains, the polydisc being the basic case [3]. The sources consulted do not describe his method.

In 1906 Friedrich Hartogs proved that compact singularities are removable for functions of more than one variable [4, 5]. This is Hartogs's extension theorem, proved in a 90-page paper in *Mathematische Annalen* 62. The simultaneous analytic continuation of all holomorphic functions appears in a second paper of 1906, "Einige Folgerungen aus der Cauchyschen Integralformel bei Funktionen mehrerer Veränderlichen", in the *Münchener Berichte* 36 [6]. The sources do not describe his method.

In 1907 Poincaré proved that the ball and the bidisc in $\mathbb{C}^2$ are not biholomorphic [7, 2, 8]. Their automorphism groups are Lie groups of different dimension. (A check: dim $\mathrm{Aut}(B^2)$ = dim PU(2,1) = 8, while dim $\mathrm{Aut}(D^2) = 2 \cdot \dim \mathrm{PSL}(2,\mathbb{R}) = 6$. The Bergman kernels confirm that the two domains look different: $2/(\pi^2(1-|z|^2)^3)$ for the ball and $1/(\pi^2(1-|z_1|^2)^2(1-|z_2|^2)^2)$ for the bidisc give 0.48034 and 0.17340 at (0.4, 0.3).) Like Hartogs's theorem, this result stood on no earlier link in the chain.

In 1911 E. E. Levi posed the Levi problem: is every pseudoconvex domain a domain of holomorphy [9, 2]? Noguchi dates his papers to 1910 and 1911 [6]. A domain whose Levi form is positive definite is called strongly Levi pseudoconvex [2]. (A check: for the defining functions $r = |z_1|^2 \pm |z_2|^2 - 1$, the Levi form at (1, 0) in the complex tangent direction (0, 1) is +1 for the ball and −1 for its indefinite counterpart.)

In 1932 Henri Cartan and Peter Thullen proved that a domain is a domain of holomorphy exactly when it is holomorphically convex [10, 11, 6]. The sources do not describe the proof. In 1934 Heinrich Behnke and Thullen published a monograph

that listed the open problems of the field [12, 6]. Noguchi names its "Three Big Problems": the Levi problem, the Cousin problems I and II, and their developments [6].

Oka and the Levi problem, 1939–1958

In 1939 Kiyoshi Oka proved what is now called the Oka principle [12, 13, 6, 14]. On a domain of holomorphy, a second Cousin problem that can be solved continuously can also be solved holomorphically. The only obstruction is therefore topological. Behnke and Thullen's list of problems steered this research [12, 6].

In 1942 Oka solved the Levi problem for n = 2, in "Domaines pseudoconvexes" [10, 13]. He had announced the result in 1941 for univalent domains of C^2, and the 1942 paper added a remark for n ≥ 2 (note 1). His tool was his Heftungslemma, a gluing lemma he proved with Weil's integral formula [6]. This solved the problem Levi had posed 31 years earlier (1942 − 1911 = 31).

In 1948 Behnke and Karl Stein proved that every non-compact connected Riemann surface is Stein, in *Mathematische Annalen* 120 [16, 2, 17]. This was independent of Oka's work.

In 1950 Oka proved that the sheaf of holomorphic functions is coherent [2, 18, 6]. The paper is Oka VII, in the *Bulletin de la SMF* 78 (1950), received in 1948. Cartan simplified the proof and introduced the term "faisceau cohérent" [6]. The sources do not describe Oka's method; Noguchi says Oka first used the preparation theorem in 1944, not in this paper.

Around 1951–53 Cartan proved Theorems A and B on Stein manifolds [19, 18, 11]. A coherent sheaf is spanned by its global sections, and its cohomology H^q vanishes for q ≥ 1. The cohomological form of Theorem B is credited to Serre; Demailly places it in Cartan's seminar of 1951–52, after discussions with Serre [11, 19]. According to Demailly, Cartan could not prove the general coherence result without the conjecture Oka proved in 1950, and Cartan's work started from Oka's Cousin papers [11] (note 2).

In 1953–54 the Levi problem was solved in all dimensions, independently [9, 2, 6]. Oka solved it in 1953 for unramified Riemann domains over C^n. Bremermann and Norguet solved it in 1954 for univalent domains (note 3). Pseudoconvex domains carry a plurisubharmonic exhaustion function [9]; the sources do not describe the methods of proof. This answered Levi's question 42 years after it was posed (1953 − 1911 = 42). No source states that these solutions used the Cartan–Thullen theorem, so that theorem counts here as a parallel result.

In 1956 Serre proved GAGA [19, 20]. On a projective variety every analytic coherent sheaf comes from a unique algebraic one, with the same cohomology. The proof compares the cohomology of Serre twists. Serre says that the proofs of §3 rest "principalement" on his earlier FAC and on Cartan's Theorems A and B, as set out in Cartan's seminar [20].

In 1958 Hans Grauert proved that a manifold is Stein exactly when it has a strictly plurisubharmonic exhaustion [15, 2]. This solved the Levi problem for manifolds. The Hilbert-space method often linked to this result belongs to Hörmander in 1965, not to Grauert [15].

The ∂̄-Neumann problem, 1963–1996

In 1963–64 J. J. Kohn solved the ∂̄-Neumann problem on strongly pseudoconvex domains [21, 22, 23]. He used the subelliptic nature of the problem to prove regularity of the Neumann operator N. The work won the Steele Prize in 1979. According to Straube, the problem had been posed by Spencer in the 1950s [21]; Kohn was Spencer's student [22]. Kohn's solution dates from 1962 (note 4).

In 1965 Lars Hörmander proved L^2 estimates and existence theorems for ∂̄ with weights, plurisubharmonic weights in Chapter II, in *Acta Mathematica* 113 [24, 21, 25]. The weak and strong extensions of the operator coincide, by Friedrichs' methods, and this lets the proof bypass boundary regularity. As the paper's introduction says, it avoids "the difficult questions of boundary regularity discussed by Kohn" and gives new proofs of the Andreotti–Grauert theorems [25]. It also gives the operator N_q on bounded pseudoconvex domains [21]. Andreotti and Vesentini found related results in parallel in 1965, which Hörmander's introduction acknowledges.

In 1974 Charles Fefferman proved that a biholomorphism between smoothly bounded strongly pseudoconvex domains extends smoothly to the closures [28, 29]. The proof uses delicate estimates of the Bergman kernel and appeared in *Inventiones Mathematicae*. No source states that it used Kohn's ∂̄-Neumann results, so the two count here as independent. (A check: the Bergman kernel summed from orthonormal monomials matches the closed form. For the disc at r = 0.6, with n below 400, the sum is 0.7771237455658951 against 0.7771237455658953. For the ball at (0.4, 0.3), with exponents below 150, both give 0.480337463193305.)

In 1980 Steven Bell and Ewa Ligocka proved smooth extension for a wide class of pseudoconvex domains, in *Inventiones Mathematicae* 57 [28, 30]. They worked through the regularity of the Bergman projection. Their work simplified and extended Fefferman's theorem [30]. According to McNeal, it rests on the global regularity of Kohn's operator [28]. The form known as Condition R is credited to Bell by McNeal and to Bedford, Bell and Catlin by Darko.

In 1983 and 1987 David Catlin proved that the ∂̄-Neumann problem is subelliptic exactly when every boundary point is of finite type [21, 23]. He used bounded plurisubharmonic weights with large Hessians (D'Angelo–Kohn, Theorem 7.2). His theorem belongs to Kohn's ∂̄-Neumann theory, on which it builds (note 5).

In 1996 Michael Christ proved that the ∂̄-Neumann operator is not globally C^∞ regular on worm domains, in the *Journal of the AMS* [21, 28]. This answered negatively the question left open after Bell–Ligocka: is N always globally regular? It is not a counterexample to the smooth extension of biholomorphisms [28].

The Oka line and hyperbolicity, 1970–2026

In 1970 Shoshichi Kobayashi conjectured that a generic hypersurface of degree d ≥ d_n in P^(n+1) is hyperbolic [26, 27]. This opened the hyperbolicity line.

In 1989 Mikhail Gromov proved an Oka principle for sections of elliptic bundles, in the *Journal of the AMS* 2, 851–897 [15, 31, 32]. A dominating spray gives the parametric Oka property for maps from Stein manifolds. As his title says, the paper generalises Oka's 1939 principle to a homotopy-theoretic form, fifty years later (1989 − 1939 = 50).

In 2011–13 Franc Forstnerič and Finnur Lárusson developed the theory of Oka manifolds [31, 33, 34]. The Oka properties known as CAP, CIP, BOP and POP are equivalent, and ellipticity is the main sufficient condition. Lárusson added a framework from model categories. The work rests on Gromov's 1989 theorem that ellipticity implies the Oka property, as Kusakabe states [34].

In 2016 Damian Brotbek proved the Kobayashi conjecture, in a preprint published in 2017 in *Publications Mathématiques de l'IHÉS* 126, 1–34 [27, 36]. He used a Wronskian construction of twisted jet differentials. Ya Deng then made the degree bound d_n explicit, "based on the work of Damian Brotbek", and Demailly improved Deng's bound [35, 26]. (A check: Demailly's bound ⌊(en)^(2n+2)/5⌋ gives 10, 5163, 3 911 613 and 4 619 284 679 for n = 1 to 4. The conjecture was 46 years old: 2016 − 1970 = 46.) Siu also contributed to this line [26].

In 2018 Yuta Kusakabe proved that Gromov's condition Ell_1 characterises Oka manifolds, answering conjectures of Gromov [34, 37]. A localisation principle for Oka manifolds follows as an application. The paper appeared in 2021 in the *Indiana University Mathematics Journal* 70.

In 2019 Gergely Bérczi and Frances Kirwan proved the Kobayashi and Green–Griffiths–Lang conjectures for generic hypersurfaces of polynomial degree [38, 39]. The method is intersection theory on non-reductive GIT quotients. The paper appeared in *Inventiones Mathematicae* in 2023 [40], 53 years after Kobayashi's conjecture (2023 − 1970 = 53). For Green–Griffiths–Lang it follows the strategy of Demailly and Siu, in the form of Diverio, Merker and Rousseau; the Kobayashi case then follows from a result of Riedl and Yang [45].

In 2024 Benoît Cadorel gave a new version of the Bérczi–Kirwan result with classical Green–Griffiths jets, using Siu's slanted vector fields and holomorphic Morse inequalities [41]. His abstract states both dependencies. The preprint had no journal reference as of 29 October 2026.

In 2024 Kusakabe proved that the complement of a compact polynomially convex set in $\mathbb{C}^n$ is Oka, in *Annals of Mathematics* 199, 899–917 [42, 37]. Andrist, Shcherbina and Wold had shown in 2016, in *Arkiv för Matematik* 54, that such complements with non-empty interior in $\mathbb{C}^n$, $n \geq 3$, are not subelliptic [37]. Together these give non-elliptic Oka manifolds, a negative answer to Gromov's question in the non-compact case [42, 37]. Earlier, in 2013, Andrist and Wold had shown that the complement of the ball in $\mathbb{C}^n$, $n \geq 3$, is not subelliptic [43].

In 2025 Forstnerič and Lárusson proved that every projective Oka manifold is elliptic, their Theorem 1.1 [44, 37]. This answers Gromov's 1989 question in the projective case, 36 years later (2025 − 1989 = 36). The proof uses Cartan's Theorem A, Grauert's 1-convexity, Kodaira's work and results of Prezelj and Stopar. Kusakabe's results serve only as background: the paper cites Kusakabe 2021 only for localisation, not his Ell_1 criterion. It was published in 2026 in *Mathematical Research Letters* (note 6). Whether a compact non-projective Oka manifold exists that is neither elliptic nor subelliptic was still open at that point.

Notes

1. The Encyclopedia of Mathematics gives Oka's 1942 result as n = 2 only; Noguchi says the 1942 paper, the full version of the 1941 announcement for $\mathbb{C}^2$, adds a

remark for all $n \geq 2$.

2. Only Demailly states Cartan's use of Oka's coherence theorem and the influence of Oka's Cousin papers on Cartan. Wikipedia dates Theorems A and B "around 1951"; there is no single date.
3. The Encyclopedia of Mathematics says Oka, Norguet and Bremermann solved the Levi problem independently, while Wikipedia says Bremermann reduced a Stein-domain result to Oka's. Noguchi settles this: Oka (1953) solved it for unramified Riemann domains over $\mathbb{C}^n$, and Bremermann and Norguet (1954), independently, for univalent domains in $\mathbb{C}^n$. The two encyclopedias describe different results.
4. Kohn solved the $\bar{\partial}$-Neumann problem in 1962 (D'Angelo–Kohn) and published in 1963–64 (Straube). Wikipedia and the Princeton page title his prize paper "strongly convex domains", against the *Annals* title "strongly pseudo-convex manifolds", probably a slip. Only Straube states that the problem was Spencer's.
5. D'Angelo–Kohn and Straube place Catlin's theorem within Kohn's theory, but neither says that the proof uses Kohn's 1979 multipliers.
6. The arXiv record gives the DOI 10.4310/MRL.260503003231 for *Mathematical Research Letters*; an earlier listing gave volume 33(1), pages 297–312, which no source consulted here confirms. The paper was posted on 27 February 2025 and the author PDF is dated 29 March 2025. Brotbek is 2016 as a preprint and 2017 in print; Bérczi–Kirwan 2019 as a preprint and 2023 in print. Andrist–Shcherbina–Wold is 2016, not 2024. The theorem of Hartogs on separately analytic functions and the Weierstrass preparation theorem are not dated in the sources used for this chapter.

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Special functions

The chain of special functions starts with Wallis's *Arithmetica infinitorum* (1655–56), which gave the infinite product for π and the word "hypergeometric". From the eighteenth century four lines grow out of it. The Γ line runs from Euler's interpolation of n! (1729–30) through Legendre, Hölder (1887) and Bohr–Mollerup (1922) to the Selberg integral (1944) and its use by Mehta–Dyson and Macdonald. The hypergeometric line runs from Gauss (1812) through Kummer and Riemann to Askey–Gasper (1976) and Askey–Wilson (1985). On its side of ordinary differential equations stand the Bessel functions and the Painlevé transcendents, which were proved irreducible only in 1988–89.

The elliptic line runs from Fagnano and Euler through Legendre to Abel (1827) and Jacobi (1829). The q-series line runs from Rogers (1894) to Schur (1917). Abel and Jacobi worked independently, and Ramanujan and Schur rediscovered Rogers's identities rather than building on them. In all there are 32 links. The newest, from 2025–26, are three unrefereed preprints, among them a proof of the 1998 q-Baker–Forrester conjecture.

Wallis, Euler and the first functions, 1655–1786

In 1655, in his *Arithmetica infinitorum* printed in 1656, John Wallis found the infinite product for π [1, 3, 4]. He got it by interpolating the integrals of $(1 - x^2)^n$ at the values n = 1/2, 3/2, … [3, 4]. The same book introduced the term "hypergeometric series" [2, 5]. (A check: the partial products of $\Pi\ 4n^2/(4n^2 - 1)$ for n = 1 to N, in 40-digit arithmetic, are 1.5338519033 for N = 10, 1.5704038730 for $N = 10^3$ and 1.5707923998 for $N = 10^5$. They approach π/2 = 1.5707963268.)

In 1729–30 Daniel Bernoulli and Leonhard Euler interpolated n!, the factorial, to non-integer values [5, 1, 6]. Bernoulli gave a product on 6 October 1729 and Euler a product on 13 October 1729. On 8 January 1730 Euler gave an integral, which in modern form is $\int_0^\infty t^x e^{-t}\,dt$ and satisfies [x+1] = (x+1)[x] [6, 1]. The memoir, E19, was submitted on 28 November 1729 and printed in 1738 [5, 6]. At x = 1/2 Euler's product reduces to Wallis's, which gives √π; this is how the work rests on Wallis [1]. Whether Euler used Bernoulli's product is not known. (Checks: Γ(1/2) = 1.772453850905516 = √π, to 40 digits; $\int_0^\infty t^4 e^{-t}\,dt = 24 = 4!$; and Γ(0.3)Γ(0.7) = 3.8832220774509331547 = π/sin(0.3π), the reflection formula. See note 1 on the value at 1/2.)

In 1732 Daniel Bernoulli studied the hanging chain and met the first function of Bessel type, J_0 [7, 8]. According to Wikipedia he also gave a method for finding its zeros (note 2). The paper was printed in 1738 in the *Commentarii* of St Petersburg, volume 6 [9]. (A check: the first zero of J_0 is 2.404825557695773, to 40 digits.)

Euler took up the chain in 1736 and again in his later papers [7, 9]. According to Wikipedia he linked Bernoulli's solution to what are now the Laguerre polynomials, found that non-uniform chains give modified Bessel functions, and treated membranes in cylindrical coordinates with power series for integer order [7]. Dutka's bibliography

lists the chain papers E49 (1736) and E576–577 (1781) [9]. The membrane work is dated 1780 by Wikipedia and 1764 by Brown [7, 8] (note 3).

Euler also gave an integral representation of the hypergeometric series ${}_2F_1$ by a beta-type integral, known as Euler's formula [10, 2]. From it follow the Euler and Pfaff transformations [2]. Wikipedia dates the integral to 1748; Kampé de Fériet cites the *Institutiones calculi integralis* of 1769, and Papadopoulos places Euler's study of these series in that treatise of 1768–70 [2, 11, 26] (note 4).

In 1750 Giulio Fagnano published an algebraic relation between lemniscatic integrals, the duplication formula [12, 14]. In 1751 Euler was asked to examine Fagnano's treatise, and from it he went on to the addition theorem for elliptic integrals [14, 13]. This began the elliptic line.

In 1782, in a paper printed in 1785, Adrien-Marie Legendre introduced the Legendre polynomials [15, 16]. They are the coefficients of $1/\sqrt{(1 - 2xt + t^2)}$, which arises in expanding the Newtonian potential of spheroids. (Checks: the coefficients of t^0 to t^5 equal P_0 to P_5; $P_4 = 35x^4/8 - 15x^2/4 + 3/8$; $\int P_3 P_5\,dx = 0$ and $\int P_4^2\,dx = 2/9$ over $[-1, 1]$; and $P_4(0.3) = {}_2F_1(-4, 5; 1; 0.35) = 0.0729375$.) That each P_n is a ${}_2F_1$ is a mathematical relation only; Legendre did not build on the hypergeometric series, so the two are independent links. The date of the polynomials varies (note 5).

Legendre, Gauss, Bessel and the elliptic functions, 1786–1836

From 1786 to 1830 Legendre built the theory of elliptic integrals [16, 12]. He wrote on integration by elliptic arcs in 1786, according to MacTutor, and classified the integrals into three kinds [16, 12]. He collected the work in the *Exercices* (1811, 1817, 1819) and in the *Traité* (1825, 1826 and 1830 per MacTutor; 1825–32 per Wikipedia) [16, 12].

Around 1811, or in 1814, Legendre named Euler's integral Γ and wrote it in its modern form [6, 17]. This builds directly on Euler's definition of 1729–30 (note 6).

In 1812, in a memoir printed in 1813, Carl Friedrich Gauss gave the first systematic theory of ${}_2F_1$ [2, 18]. He proved the summation formula ${}_2F_1(a, b; c; 1) = \Gamma(c)\Gamma(c - a - b)/(\Gamma(c - a)\Gamma(c - b))$ and gave the contiguous relations, "fifteen in number" [2, 19, 10, 18]. The sources do not describe how Gauss himself derived the sum. Today it follows from Euler's integral at $z = 1$ [2], but no source shows that Gauss used it (note 7). (A check: ${}_2F_1(0.3, 0.4; 1.5; 1) = 1.181191851094815769369215235 13$, equal to Gauss's Γ ratio to 30 digits.)

In 1817 Friedrich Wilhelm Bessel introduced the Bessel functions from the Kepler problem [20, 7]. He worked by the method of Fourier series, which he introduced independently, and gave Bessel's integral [7]. The systematic study followed in 1824 [20, 7]. He drew on special cases found by Bernoulli, Euler and Lagrange as influence rather than as steps of proof, and simplified Lagrange's series for the Kepler problem [7, 9]. The dates vary (note 8). (A check: Bessel's integral $(1/\pi)\int_0^{\pi} \cos(n\theta - x \sin\theta)\,d\theta$ equals the power series to 20 digits: $J_0(1.5) = 0.51182767173591812875$, $J_1(2) = 0.57672480775687338720$ and $J_3(5) = 0.36483123061366699446$.)

In 1827–28 Niels Henrik Abel inverted the elliptic integral of the first kind and obtained doubly periodic elliptic functions, in *Crelle's Journal* [13, 22]. The second part, of 1828, derives Jacobi's transformations in an appendix [13, 21]. Abel's starting

point was Legendre’s work: he begins by recalling Euler and Legendre and reduces to Legendre’s three kinds before inverting the first [13, 22]. Jacobi’s results also follow from the addition theorems that Euler had proved, and MacTutor calls Abel’s theorem “a vast generalisation of Euler’s relation” [13, 21] (note 9). The appendix re-derives Jacobi’s transformations of 1827. (A check: at m = 0.49 and 35-digit precision, sn(F(0.9)) = 0.78332690962748338846 = sin 0.9, and sn(u + 4K) − sn(u) and sn(u + 2iK′) − sn(u) vanish to working precision, below 10^{-35}.)

In 1829 Carl Gustav Jacob Jacobi published the *Fundamenta nova* [23, 24]. It contains the elliptic functions now written sn, cn and dn, a notation due to Gudermann and Glaisher, and the four theta functions [24, 23]. The sources do not say that he reached them through his transformation theory. Abel and Jacobi worked “quite independently and at much the same time”, as MacTutor puts it, and Jacobi proved his results “without referring to Abel” [23, 13] (note 10). (Checks at u = 0.83, m = 0.49: $\mathrm{sn}^2 + \mathrm{cn}^2 = 1$ and $\mathrm{dn}^2 + m\,\mathrm{sn}^2 = 1$; and K(0.49) = 1.8456939983747235176, to 35 digits.)

In 1836 Ernst Kummer found the 24 solutions of the hypergeometric equation and the first quadratic transformations [2, 10]. He used changes of variable that permute the singular points 0, 1 and ∞. His work extends Gauss’s theory of the same series (note 11).

The Γ function characterised, 1856–1922

In 1856 Karl Weierstrass characterised Γ by f(1) = 1, f(x + 1) = x f(x) and a limit condition [1, 25]. Aycock places this in the 1856 paper; Iurato says it was observed in 1854. The 1856 memoir on analytical factorials works with the product for 1/Γ, which had been found earlier. Aycock credits Schlömilch (1844); Iurato credits Schlömilch (1843) with the product for Γ and F. W. Newman (1848) with that for 1/Γ. Weierstrass’s general factorisation theorem followed in 1876 according to Iurato, 1878 according to Aycock (note 12).

In 1887 Otto Hölder proved that Γ satisfies no algebraic differential equation [6, 1]. According to Wikipedia, the idea is that a solution of such an equation could not also satisfy Γ(x + 1) = xΓ(x) [6]. The theorem proves a property of Euler’s function.

In 1922 Harald Bohr and Johannes Mollerup proved that Γ is the only positive function f on x > 0 with f(1) = 1, f(x + 1) = x f(x) and log f convex [29, 1]. Log-convexity squeezes f between bounds whose limit is an explicit product formula [29]. This characterises Euler’s Γ. (Checks: the smallest second difference of log Γ over 2000 equally spaced points of [0.05, 10] is 2.607×10^{-6}, positive, so log Γ is convex there. The Gauss limit $n^x\, n!/(x(x+1)\dots(x+n))$ at x = 0.5 gives 1.7723874 for $n = 10^4$ and 1.7724532 for $n = 10^6$, tending to $\sqrt{\pi} = 1.7724538509$.)

Hypergeometric equations, q-series and Painlevé, 1857–1917

In 1857 Bernhard Riemann characterised ${}_2F_1$ by its differential equation, with three regular singular points on the sphere [2, 27]. The solutions are now called the Riemann P-function; the general form of the equation is credited to Papperitz [27]. This extends the work of Gauss and Kummer. Riemann’s announcement of 1857 names the works of Wallis, Euler, Pfaff, Gauss and Kummer [2, 26].

In 1873 H. A. Schwarz gave his list of the 15 algebraic cases of ${}_2F_1$, those with finite monodromy, in *Crelle* 75 [28, 11]. In 1881 Édouard Goursat gave a complete list of the quadratic transformations and the first cubic ones, in the *Annales de l'ÉNS* 10 [2, 11]. The count of 15 and the content of Goursat's list rest on Wikipedia, while Kampé de Fériet's bibliography confirms both papers and their years.

In 1894 Leonard James Rogers first proved the Rogers–Ramanujan identities [30, 31, 32]. Ramanujan rediscovered them, without proof, before 1913. Issai Schur rediscovered and proved them independently in 1917. These were rediscoveries, not steps built on Rogers. Schur wrote that Frobenius had shown him a direct proof of the polynomial identities that imply them [33]. In 1919 a proof by Rogers and Ramanujan followed, 25 years after Rogers's first (1919 − 1894 = 25) (note 13). The sources do not describe the methods of proof. (A check: $\Sigma\, q\hat{}(n^2)/(q;\,q)_n$ equals the product of $1/(1 - q\hat{}m)$ over $m \equiv \pm 1 \pmod 5$, coefficient by coefficient through q^{59}, in exact integer arithmetic.)

In 1900 and 1902 Paul Painlevé classified the second-order equations $y'' = R(x, y, y')$ without movable critical points [34, 35]. He found 50 canonical forms and six new transcendents. The sources do not describe his method. In 1889 Émile Picard had already found a solution of a special case of the sixth equation, with all parameters zero, before the equation itself was known [49] (note 14).

In 1905 Richard Fuchs found the general sixth Painlevé equation, P6, as the isomonodromy equation of a Fuchsian equation with four singular points [34, 36]. He did so independently of Painlevé, through isomonodromy, according to Wikipedia [34].

In 1910 Bertrand Gambier corrected and completed Painlevé's classification and added the general P6 to the list [34, 35]. This repairs the errors of Painlevé's work, ten years after it (1910 − 1900 = 10). Whether Gambier took P6 from Fuchs is not known.

Integrals, polynomials and the newest links, 1944–2026

In 1944 Atle Selberg evaluated what is now the Selberg integral, an n-dimensional Euler beta integral, as a product of Γ ratios, in *Norsk Matematisk Tidsskrift* 26, 71–78 [37, 38]. He proved it for integer γ by expanding the product and using Euler's beta integral, then continued it analytically by an argument of Carlson's type [38]. It generalises Euler's beta integral. (Checks: $S_2(2, 3, 1) = 1/1800$ exactly, by direct integration and by the formula; $S_2(1.5, 2.5, 0.5) = 0.00952380952381 = 1/105$ by the formula and by numerical quadrature; and $S_1(2, 3) = B(2, 3) = 0.0833333$.)

In 1963 Madan Lal Mehta and Freeman Dyson conjectured the Gaussian integral now named after Mehta, without knowing Selberg's work [37, 38]. It was later proved as a limit of the Selberg integral, nineteen years after Selberg's paper (1963 − 1944 = 19).

In 1976 Richard Askey and George Gasper proved that the partial sums $\Sigma_{k=0}^{n}$ P_k^(α,0)(x) of Jacobi polynomials are non-negative on [−1, 1], in "Positive Jacobi polynomial sums. II", *American Journal of Mathematics* 98, 709–737 [39, 40, 41]. They had found the result in 1973 [41]. De Branges's proof of the Bieberbach conjecture (1984, printed 1985; see the chapter on functions of a complex variable) reduced to this inequality. As MacTutor puts it, the problem "ultimately revealed

itself to be an inequality established several years earlier by Askey and Gasper” [40, 39, 41]. The result rests on the theory of Jacobi polynomials (note 15). (A check: on 201 equally spaced points of $[-1, 1]$, for n up to 20 and α in $\{0, 1, 2, 4, 6\}$, the smallest value is 0, reached at $x = -1$ for odd n; numerically -6.7×10^{-16}, which is rounding.)

In 1982 Ian Macdonald conjectured Mehta’s integral for all finite root systems [38, 37]. In 1989 Eric Opdam gave a uniform proof for the crystallographic groups, using Heckman–Opdam shift operators. The groups H_3 and H_4 were done by Opdam and by Garvan, partly by computer, which gave full generality [38, 37]. The line rests on Selberg’s 1944 integral, which had proved cases.

In 1985 Askey and James Wilson introduced the Askey–Wilson polynomials, defined by a ${}_4\varphi_3$ basic hypergeometric series [42, 43]. Many other orthogonal polynomials of the Askey scheme are special or limiting cases of them [42]. The memoir, *Memoirs of the AMS* 54, no. 319, is titled “Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials” and gives their orthogonality relation [43]. They generalise the Jacobi polynomials and are q-analogues of the Wilson polynomials [51].

In 1988 Keiji Nishioka and in 1989 Hiroshi Umemura proved that the Painlevé equations do not reduce to classical functions, by differential-algebraic irreducibility [34, 36, 44]. Umemura did this for P_I, and gave a second proof in *Nagoya Mathematical Journal* 117 [36, 44]. A series of papers by Okamoto, Nishioka, Noumi, Umemura, Watanabe and others extended irreducibility to all six equations, outside special values of the parameters [50]. This proved Painlevé’s claim, made 87 years earlier, that they define new functions ($1989 - 1902 = 87$).

In 2025 Wang, Yang and Zhao posted a preprint (v1, 6 April 2025) proving an absolute-monotonicity result for zero-balanced ${}_2F_1$ [45]. By a recurrence method it answers the 1997 problem of Anderson, Vamanamurthy and Vuorinen on Ramanujan’s asymptotic formula, 28 years after it was posed ($2025 - 1997 = 28$). The preprint had no journal reference as of 29 October 2026 (note 16).

In 2025 Yue Zhou posted a preprint (v1, 24 March 2025) proving the $(p + 1)$-component q-Baker–Forrester conjecture of 1998 [46]. The conjecture generalises the q-Morris identity and hence the Selberg integral; the case $p = 1$ had been proved in 2015. The proof uses the Gessel–Xin constant-term method with a splitting formula. The preprint had no journal reference as of 29 October 2026 (note 16).

In 2026 two preprints reclassified the polynomials of the Askey scheme [47, 48]. Luis Verde-Star (June 2026) found 53 disjoint classes of hypergeometric orthogonal polynomials “different from the Askey scheme”. Castillo and Gordillo-Núñez (March 2026) classified the classical ones on linear lattices in Maroni’s dual framework (note 16).

Notes

1. The Euler Archive summary of E19 says $[1/2] = \sqrt{(\pi/2)}$. The recomputed value is $(1/2)! = \sqrt{\pi}/2 = 0.8862$, not $\sqrt{(\pi/2)} = 1.2533$, and Aycock agrees with the computation.
2. Only Wikipedia credits Bernoulli with a method for the zeros; the other sources credit the power series to Euler, not to Bernoulli. Bernoulli’s chain is dated 1732 (Wikipedia, Brown) and printed 1738 (Dutka), which is compatible.

3. Only Wikipedia states Euler's link to the Laguerre polynomials. Euler's membrane work is 1780 per Wikipedia and 1764 per Brown; not settled.
4. Euler's integral for ${}_2F_1$ is dated 1748 by Wikipedia, while Kampé de Fériet cites the *Institutiones* of 1769; not settled. The Euler and Pfaff transformations as consequences are stated only by Wikipedia.
5. Wikipedia dates the Legendre polynomials to 1782 (printed 1785) and MacTutor to a paper of 1784; not settled.
6. Wikipedia dates Legendre's symbol Γ "around 1811", the Encyclopedia of Mathematics 1814, and Aycock cites it from a Legendre work of 1826; not settled.
7. Gauss's memoir is dated 1812 (Aycock, Wikisource) or 1813 (Wikipedia): probably presented in 1812 and printed in 1813.
8. MacTutor and Brown date Bessel's functions to 1817; Wikipedia gives a letter of 1813 published in 1819; Dutka's bibliography lists the Kepler paper as 1816–17, printed 1819. Most likely the work was done in 1816–17 and printed in 1819.
9. MacTutor's sentence about "a vast generalisation of Euler's relation" refers to Abel's theorem, not specifically to this paper.
10. Abel published first. The link between Jacobi's 1827 transformations and Abel's 1828 appendix runs from Jacobi to Abel, not from the 1829 *Fundamenta nova*.
11. The extension of Gauss by Kummer is inferred from the content: Kummer's paper, "Über die hypergeometrische Reihe", treats Gauss's series; no source consulted states the dependency in words.
12. Weierstrass's characterisation is placed in 1856 by Aycock and in 1854 by Iurato; the earlier product is Schlömilch's for $1/\Gamma$ (1844) per Aycock, while Iurato gives Schlömilch (1843) the product for Γ and Newman (1848) that for $1/\Gamma$; the factorisation theorem is 1878 per Aycock and 1876 per Iurato. None is settled. The Encyclopedia of Mathematics and Wikipedia give the product without Weierstrass's 1856 date.
13. The 1919 proof is joint per Wikipedia and Rogers's alone per MathWorld; not settled. Only Doyle reports what Schur wrote about Frobenius.
14. The sixth Painlevé equation has no single discoverer: Picard found a special case (1889), Fuchs the general equation (1905), and Gambier added it to the list (1910). That Fuchs worked independently of Painlevé rests on Wikipedia; nLab lists Fuchs's 1905 note and the isomonodromy connection but does not say this.
15. The exact form of the Askey–Gasper sum is not legible in the sources consulted and rests on the recomputation; with the normalisation P_k^(0,α)(1) = 1, the sum shown is the standard form.
16. The 2025–26 results are preprints, each supported only by its authors' own paper as of 29 October 2026. Zhou's dependence on the Selberg integral and the two reclassifications of the Askey scheme are as the papers state.

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Difference and functional equations

The chain of difference and functional equations starts with Indian metre counting and Fibonacci's recurrence, which de Moivre solved in closed form. It then runs in three lines. The first concerns linear recurrences: the Skolem–Mahler–Lech theorem and the Skolem problem. The second is Cauchy's functional equation and the Hyers–Ulam stability that grew from it. The third is one-dimensional discrete dynamics, from Sharkovskii to Lanford.

In all there are 23 links, from Virahanka to a survey of the Skolem problem in 2026. As of 2026 the Skolem problem is decidable up to order 4 (Bacik, published December 2025) and open at order 5. Three pairs of results were found independently rather than built on each other: the Indian rule and Fibonacci, de Moivre and Binet, and Sharkovskii and Li–Yorke.

Recurrences from metre to closed form, 6th century–1843

Between the sixth and eighth centuries Virahanka, later Gopala, and around 1150 Hemachandra counted the poetic metres of total length n [1, 2]. A metre ends either in a short syllable or in a long one. So the count for each length is the sum of the counts for the two previous lengths. This rule stood on nothing earlier in the chain.

In 1202 Fibonacci posed the rabbit problem in his *Liber abaci* [3, 2]. The pairs this month are last month's pairs plus the newborn pairs, so $F(n + 2) = F(n + 1) + F(n)$. (A check: F(1) to F(13) are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233.) Singh dates the Indian rule earlier, but neither source claims that Fibonacci knew it, so the two are independent discoveries [1, 3].

In 1718 Abraham de Moivre found the closed form $F(n) = (\varphi^n - \psi^n)/\sqrt{5}$, where φ and ψ are the roots of $x^2 = x + 1$ [4, 2]. In 1843 Jacques Binet found it again, and it now bears his name [4]. Both rest on Fibonacci's recurrence; Binet's was a rediscovery, not a step built on de Moivre (note 1). The sources do not describe how de Moivre derived it. (A check: Binet's formula agrees with the recurrence for n = 0 to 100 in 60-digit arithmetic. In ordinary double-precision arithmetic the rounded value first fails at n = 71.)

Cauchy's equation and its stability, 1821–1991

In 1821 Augustin-Louis Cauchy proved that every continuous solution of $f(x + y) = f(x) + f(y)$ is $f(x) = cx$ [5, 6]. He first showed that $f(q) = q \cdot f(1)$ for every rational q, then extended to all reals by continuity. This began the line of functional equations.

In 1875 Gaston Darboux showed that continuity at a single point already forces linearity [5, 6]. Additivity carries continuity from that point to every point. This weakens the hypothesis of Cauchy's 1821 theorem.

In 1905 Georg Hamel showed that non-linear additive functions exist [7, 5]. Choose a basis of $\mathbb{R}$ over $\mathbb{Q}$, a Hamel basis, and assign values freely on it. No explicit

construction is known, since the argument uses the axiom of choice (MSC 03E25). This is a counterexample to Cauchy's theorem when no regularity is assumed, not a step built on its proof.

In 1940 Stanisław Ulam asked whether an approximate homomorphism must be close to an exact one [11, 12]. This was a question, not a proof.

In 1941 Donald Hyers answered it for additive maps between Banach spaces, in *PNAS* 27, 222–224 [11, 12]. If $\|g(x + y) - g(x) - g(y)\| \le \varepsilon$ for all x and y, there is a unique additive A with $\|g - A\| \le \varepsilon$. His direct method defines $A(x) = \lim 2^{(-n)}\, g(2^n x)$. (A check: for $g = 2x + 0.1 \sin x$, whose Cauchy difference is at most 0.26, within the bound 0.3, the values $2^{(-n)}\, g(2^n \cdot 1.7)$ are 3.4000342 at n = 10 and 3.400000000000 at n = 40. So A(x) = 2x, and $|g - A|$ at x = 1.7 is 0.0992, with supremum 0.1, inside Hyers's bound.)

In 1950 Tosio Aoki gave a version for additive maps whose Cauchy difference may be unbounded, without assuming continuity, in the *Journal of the Mathematical Society of Japan* 2, 64–66 [12, 13]. He generalised Hyers's result [13]. According to one study, his control was the weight $\varepsilon(\|x\|^p + \|y\|^p)$ [15]. Bouikhalene, Elqorachi and Rassias point out that his claim of a unique linear mapping was flawed for lack of a continuity assumption [14].

In 1978 Themistocles Rassias proved the stability of the linear mapping in Banach spaces when the Cauchy difference may be unbounded, in *PAMS* 72, 297–300 [26, 11, 13]. He used the direct method with the weight $\varepsilon(\|x\|^p + \|y\|^p)$, for p from 0 up to but not including 1. The paper cites Hyers and generalises his result [13, 26] (note 2). The result was later extended to every $p \neq 1$ [11]. In 1991 Gajda treated p above 1 and showed that stability fails at p = 1 [15].

Linear recurrences and the Skolem problem, 1934–1985

In 1934 Thoralf Skolem proved that the zero set of a rational linear recurrence is a finite set together with finitely many residue classes [8, 9]. Kurt Mahler extended this to algebraic recurrences in 1935, and Christer Lech to characteristic 0 in 1953. The theorem is now called Skolem–Mahler–Lech. The proof writes $u_n = \langle A^n v, w \rangle$. On each residue class u_n is a p-adic analytic function of n, and an analytic function with infinitely many zeros vanishes identically [9]. The proof is ineffective: it gives no bound on the zeros. The theorem studies the same objects as Fibonacci and de Moivre, but no source says the proof uses de Moivre's closed form; Tao's sketch uses the matrix form (note 3). (A check: the recurrence $u(n + 3) = 2u(n + 2) - 4u(n + 1) + 4u(n)$ with u = 0, 0, 1 has zeros at exactly {0, 1, 4, 6, 13, 52} for n = 0 to 2999, in exact integer arithmetic.)

The ineffective proof posed the Skolem problem: decide whether a given linear recurrence has any zero at all [9, 10]. It is open in general. (A check: from Skolem's 1934 paper to 2026 is 92 years. Bacik's 2025 abstract calls the problem open for "almost a century", and a 2022 paper calls the theorem "almost 90 years old", where 1934 to 2022 is 88 years [10].)

In 1953 Lech also showed that the theorem fails in positive characteristic [16, 9, 17, 18]. His counterexample is $(1 + t)^n - t^n - 1$ over $F_p(t)$, whose zeros are $n = 1, p, p^2, \ldots$. In 2005, in a paper printed in *Inventiones Mathematicae* 168 (2007), Harm Derksen proved a correct analogue. In positive characteristic the zero set is described by finite

automata: it is p-automatic [16, 18]. (A check: $(1 + t)^n - t^n - 1$ vanishes over F_2, F_3 and F_5 exactly at $n = 1, p, p^2, \ldots$ for $n = 1$ to 129, computing with polynomials mod p.)

In the mid-1980s Mignotte, Shorey and Tijdeman, and independently Vereshchagin, proved that the Skolem problem is decidable at order 3, and at order 4 for real algebraic recurrences [30, 31, 32]. They used Baker's theory of linear forms in logarithms. Mignotte, Shorey and Tijdeman published in *Crelle* in 1984 and Vereshchagin in *Matematicheskie Zametki* in 1985 [32]. Both partly solve the problem that Skolem's theorem posed.

Discrete dynamics, 1964–1999

In 1964 Oleksandr Sharkovskii proved his theorem on the periods of continuous maps of an interval, in the *Ukrainian Mathematical Journal* 16, 61–71 [19, 20]. Order the positive integers as $3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2\cdot3 \rhd 2\cdot5 \rhd \ldots \rhd 4\cdot3 \rhd \ldots \rhd 8 \rhd 4 \rhd 2 \rhd 1$. A map with a point of some period has points of every period later in the order, and every tail of the order occurs for some map. The proof uses covering relations between intervals, from the intermediate value theorem, and induction on the power of 2. (A check: the order on 1 to 40 is 3, 5, 7, …, 39, then 6, 10, …, 38, then 12, 20, 28, 36, then 24, 40, and finally 32, 16, 8, 4, 2, 1.)

In 1975 Tien-Yien Li and James Yorke proved that "period three implies chaos", in the *American Mathematical Monthly* 82, 985–992 [19, 20]. Period 3 gives every period, and also an uncountable set of chaotic points. Li and Yorke did not know of Sharkovskii's work, and their statement about periods is a special case of his [19, 21]. The two results are therefore independent. (A check: for the logistic map $x \mapsto rx(1 - x)$ at $r = 3.8315$, just inside the period-3 window that opens at $1 + \sqrt{8} = 3.8284271247$, the roots of $f^n(x) = x$ found by sign changes on 200 001 equally spaced points of [0, 1] and refined by bisection include every least period from 1 to 8. A grid count of this kind is only a lower bound.)

In 1977 Peter Štefan gave a new proof of Sharkovskii's theorem using what are now called Štefan cycles, in *Communications in Mathematical Physics* 54, 237–248 [19, 22, 23]. The paper names Sharkovskii's theorem in its title. It reproves that theorem rather than using it as a step.

In 1975 Mitchell Feigenbaum discovered the universality of period doubling, with an HP-65 calculator, and published it in 1978 and 1979 [24, 25]. The ratios of successive intervals between period doublings tend to $\delta \approx 4.6692$, and the scaling of the map tends to $\alpha \approx 2.5029$. He explained this universality by a fixed point of a renormalisation operator, which composes the map with itself and rescales it [40]. (A check: with R_k the superstable parameter of period 2^k of $x \mapsto rx(1 - x)$, found by Newton's method in 60-digit arithmetic, the ratio $\delta_k = (R_{k-1} - R_{k-2})/(R_k - R_{k-1})$ is 4.6692011 at $k = 10$.) This line started independently of Sharkovskii's.

In 1982 Oscar Lanford gave a computer-assisted proof of the Feigenbaum conjectures, in the *Bulletin of the AMS* 6, 427–434 [27, 25]. That was four years after Feigenbaum's first publication ($1982 - 1978 = 4$). The sources do not describe how Lanford's bounds were computed. Eckmann and Wittwer later filled a missing transversality step, "a small correction" [28, 25] (note 4). In 1999 Mikhail Lyubich gave the first proof without a computer, in the *Annals of Mathematics* 149, 319–420 [28, 29].

The Skolem problem at order 4 and beyond, 2024–2026

In 2024 Piotr Bacik posted a proof that the Skolem problem is decidable for all algebraic linear recurrences of order at most 4. It was published in *TheoretiCS* 4 on 2 December 2025 (DOI 10.46298/theoretics.25.28) [31, 33]. The results of the 1980s needed the sequence to be real. Bacik's lemma supplies an absolute value with the required pattern of dominant roots, and so removes that need, as Bilu explains [32].

In April 2025 Bacik, Ouaknine, Purser and Worrell posted algorithms for the p-adic zeros of a linear recurrence (arXiv 2504.14413) [34, 35]. The output is always correct, and termination assumes the p-adic Schanuel conjecture. The paper was refereed for STACS 2026 (LIPIcs 364, paper 8, 25 February 2026). It answers the ineffectiveness of the classical Skolem–Mahler–Lech proof by computing p-adic zeros, which the authors present as progress short of the Skolem problem itself (note 5).

In July 2025 Bacik, Ouaknine and Worrell showed that the bounded Skolem problem is in randomised polynomial time for each fixed order, and that the Skolem problem at order at most 4 lies in coRP, where before it was known only to lie in NP^RP [36, 37]. p-adic analysis isolates polynomially many candidate zeros, which are then tested by arithmetic-circuit identity testing. The paper appeared in SODA 2026, pages 5255–5269. It places Bacik's order-4 decision in coRP (note 5).

A survey by Bacik, Karimov, Luca, Nieuwveld, Ouaknine, Purser and Worrell, dated 26 July 2026, records the status of the problem [30, 10, 32]. It is decidable at order at most 4 and open at order 5. For simple recurrences it is decidable under the Exponential Local-Global Principle and the weak p-adic Schanuel conjecture, and up to order 7 under the first of these alone. The survey had not been published as of 29 October 2026. Bilu confirms decidability at order at most 4 independently [32].

Notes

1. Only MacTutor dates the closed form to de Moivre (1718) and Binet (1843); Wikipedia names de Moivre and Daniel Bernoulli without dates.
2. Only the paper's own reference list and Sadeghi's study state that Rassias's proof generalises Hyers's. Gajda's counterexample at p = 1 and Aoki's weight are stated only by arXiv 1407.3718.
3. Skolem's paper is dated 1934 by the 2026 survey and 1935 by Wikipedia. The 8th Scandinavian Congress met in Stockholm on 14–18 August 1934, and its proceedings were printed in Lund in 1935 [38]; arXiv 2308.01152 cites the paper as 1934 [39].
4. Wikipedia alone dates the Eckmann–Wittwer correction to 1987.
5. The links of the two 2025 papers to earlier work are stated by the papers and by the 2026 survey, whose author group overlaps with theirs. The survey describes the p-adic procedure without naming its authors.
6. Several parts of the field are not traced in this chapter: Birkhoff (1911), Nörlund (1924), the Poincaré–Perron theorem, Aczél's work, Hilbert's fifth problem and q-difference equations. No result from 2025–26 on the functional-equation side was found. MSC 39A covers difference equations and 39B functional equations and inequalities, with 39B82 for stability.

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Series and summability

This field asks when an infinite sum has a value, what that value is, and how to give a value to sums that do not converge in the ordinary sense. Its chain has three roots: Archimedes' geometric series, Oresme's proof that the harmonic series diverges, and the Kerala series for the arctangent. Then comes the Basel problem, which Mengoli could not solve and Euler solved through a product for the sine that he did not justify.

From there two lines run. One is rearrangement: Dirichlet and Cauchy, then Riemann, then Lévy and Steinitz. The other is summability with its Tauberian converses: Abel, Cesàro and Borel on one side; Tauber, Littlewood, Hardy and Littlewood, Karamata, Wiener and Newman on the other. In 2026 the Tauberian line is active in number theory through a guide by Pierce, Turnage-Butterbaugh and Zaman.

The links are mostly converses, generalisations and new proofs. No source says that a later proof in this chain uses an earlier result as a step, so every dependency below is one of content, not of proof. The Kerala series did not reach Gregory and Leibniz: they rediscovered it. Archimedes, Oresme, Mādhava, Borel and the Silverman–Toeplitz theorem stand without a stated link into the rest of the chain.

Roots: geometric series, the harmonic series and the Basel problem

In the third century BC Archimedes proved that a segment of a parabola has 4/3 the area of its inscribed triangle [1, 2]. He filled the segment by exhaustion with triangles whose areas form a finite geometric series of ratio 1/4, then closed the argument by a double reductio ad absurdum. (A quick check: the first 30 terms of Σ 4^−k give 1.333333333333333, and 4/3 minus this partial sum is 1.156 × 10^−18, which is exactly 4^−29/3.) Nothing later in the chain is said to rest on it.

Around 1350 Nicole Oresme proved that the harmonic series 1 + 1/2 + 1/3 + … diverges [3, 4, 5]. He grouped the terms in blocks of 2, 4, 8 and so on. Each block adds at least ½, so the partial sum up to 2^k is at least 1 + k/2. (Check: the sum up to 32 is 4.058495, above 3.5 for k = 5; the sum up to 2^20 is 14.440160, above 11 for k = 20.)

Around 1400 Mādhava or his school in Kerala found the arctangent series and with it π/4 = 1 − 1/3 + 1/5 − … [6, 7]. It is known through later Kerala texts such as the Yuktibhāṣā. These texts also give three corrections for the remainder after n terms, among them n/(4n² + 1) [6, 8]. How the series was derived is not stated in the sources; MacTutor suggests term-by-term integration [6]. (Check: with 100 terms, 4 times the partial sum misses π by 1.00 × 10^−2; adding the correction $(-1)^n$ n/(4n² + 1) cuts the error to 2.50 × 10^−11.) Gregory in 1671 and Leibniz in 1673 found the same series independently; there is no line from Kerala to them (note 5).

In 1650 Pietro Mengoli proved again that the harmonic series diverges, and that the alternating harmonic series 1 − 1/2 + 1/3 − … sums to log 2 [9, 5]. He failed to sum Σ 1/n² [9, 10]. His divergence proof grouped the terms in threes and argued by

contradiction (note 6). He had first considered that sum in 1644 and published it in 1650 in his *Novae quadraturae arithmeticae* [14].

Euler solved the Basel problem: $\Sigma\ 1/n^2 = \pi^2/6$ [11, 12]. He found it in 1734 and presented it in 1735, and the paper (E41) was printed in 1740 (note 1). He treated sin s / s as a polynomial with roots $\pm k\pi$, wrote it as a product over those roots, and compared the coefficients of s^2 in the two forms [12, 10]. In the same paper he reached $\zeta(2n)$ up to exponent 12 [12]. Euler did not check that his infinite product was valid; that was justified later by the Weierstrass factorisation theorem [10, 13]. (Check: $\zeta(2) = 1.64493406684823 = \pi^2/6$, and $\zeta(4) - \pi^4/90$ is about $2 \times 10\text{^}{-31}$, zero to working precision.) The result answers the problem Mengoli posed, and Euler also had Jacob Bernoulli's 1689 bound, which showed the sum lies below 2 [10, 14].

Rearrangement: from Dirichlet to Lévy and Steinitz

Rearranging a series that converges, but not absolutely, can change its sum. Castejón, Corbacho and Tarieladze credit this to Dirichlet in 1827 and to Cauchy in 1833, in his *Résumés analytiques* [17]; Wikipedia credits Cauchy in 1833 [18] (note 2). How each of them showed it is not described in the sources. In 1837 Dirichlet proved the other half: an absolutely convergent series keeps its sum under every rearrangement [17, 18].

Bernhard Riemann went much further in his habilitation paper, submitted in 1854 and printed in 1867 or 1868 (note 3). A series that stops converging when all its terms are made positive can be rearranged to converge to any value C, in his words "einen beliebig gegebenen Werth C" [19, 18]. The rule is simple: take positive terms until the sum passes above C, then negative terms until it falls below C, and repeat. (Check: rearranging $1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots$ by this rule towards 1.5 gives 1.499992 after 200 000 terms.) Riemann credits Dirichlet with the distinction between the two classes of series, citing Crelle volume IV of 1829 [19]. Only Riemann's own text states that debt. Cauchy's 1833 observation came earlier as well, but Riemann does not credit him.

In 1905 Paul Lévy and in 1913 Ernst Steinitz extended the theorem to vectors [27, 28]. In $\mathbb{R}^n$ the set of sums reached by rearranging a series is either empty or an affine subspace. Lévy's proof was incomplete. Steinitz gave the first complete one, eight years later, through what is now the Steinitz lemma on ordering vectors that sum to zero. Bárány notes that for dimension 1 the result "is equivalent to Riemann's result" [27]. So the theorem contains Riemann's as its one-dimensional case, though no source says the proof uses it.

Summability methods: Abel, Cesàro, Borel and Toeplitz

In 1826 Niels Henrik Abel proved his continuity theorem in the first volume of Crelle's journal, pages 311–339 [15, 16]. If Σa_n converges, then $\Sigma a_n x^n$ tends to Σa_n as x tends to 1 from below. In the complex case the limit holds inside a Stolz angle, or in the Encyclopedia of Mathematics' terms, within closed triangles with a vertex at the boundary point [15]. The proof uses summation by parts, a detail only Wikipedia states [16]. (Check: the alternating harmonic series has Abel function $\ln(1 + x)$, which is 0.693097 at $x = 0.9999$, against $\ln 2 = 0.693147$.)

In 1890 Ernesto Cesàro introduced the (C,k) means of the partial sums, in the *Bulletin des sciences mathématiques* volume 14 [20, 21]. The (C,1) mean is the plain arithmetic

mean of the partial sums, and the methods are totally regular for k ≥ 0 [20]. (Check, with 20 000 terms: Grandi's series 1 − 1 + 1 − … has (C,1) mean 0.500000. For 1 − 2 + 3 − … the (C,1) mean is 0.0 at an even number of terms and 0.5 at an odd number, so it does not settle; the (C,2) mean is 0.250012, and the Abel mean at x = 0.9999 is 0.250025, both close to ¼.) Every Cesàro-summable series is also Abel-summable to the same value, and Abel's method is stronger than all the Cesàro methods [22]. That is a later comparison theorem, not something Cesàro's definition used; Abel is background here, not a source of the result.

In 1899 Émile Borel introduced his B and B′ summation methods [25, 26]. Each term a_k is replaced by $a_k t^k/k!$, which gives a series that converges, and the change is undone by a Laplace integral [25]. The method is regular [26], and B′ sums a power series inside its Borel polygon [25]. (Check: the Borel–Laplace integral gives 0.25 = 1/(1 + 3) for $\Sigma(-3)^n$, and 0.596347362323194 = $e \cdot E_1(1)$ for the wildly divergent $\Sigma(-1)^n n!$.)

In 1911 Otto Toeplitz gave necessary and sufficient conditions for a matrix summation method (a_{nk}) to be regular, that is, to keep the value of every convergent series (note 4). The absolute row sums must be bounded, every column must tend to 0, and the row sums must tend to 1 [29, 30, 31, 32]. Toeplitz proved this for triangular matrices; Louis Silverman's 1913 thesis obtained part of it. The result stands on its own in this chain.

Tauberian converses: from Tauber to the prime number theorem

Abel's theorem has no plain converse. (Check: Grandi's series is Abel-summable to ½ yet divergent, so some extra condition is needed.) In 1897 Alfred Tauber found one: if $\Sigma a_n x^n$ tends to s as x tends to 1 and $a_n = o(1/n)$, then $\Sigma a_n = s$ [23, 24]. How Tauber proved it is not described in the sources. The theorem is the converse of Abel's under a condition, not a proof that uses Abel's.

In 1911 John Edensor Littlewood showed that the weaker condition $a_n = O(1/n)$ is enough, in the *Proceedings of the London Mathematical Society* (2) 9, pages 434–448 [33, 24, 34]. His proof was much harder than Tauber's [33]; its method is not described in the sources. (Check: the terms (−1)^(n+1)/n meet $n \cdot a_n = O(1)$ but not o(1), so the new condition is truly weaker.) The result strengthens Tauber's (note 7).

In 1914 G. H. Hardy and Littlewood proved the Tauberian theorem for non-negative coefficients, in the same journal, (2) 13, pages 174–191 [23, 24, 35, 36]. It passes from Abel means to partial sums in asymptotic form, and it fails without non-negativity [23]. Borwein describes it as a generalisation of Tauber's theorem with a one-sided condition (note 8) [35]. It generalises Littlewood's 1911 result rather than using it as a step.

In 1930 Jovan Karamata gave a short proof of the Hardy–Littlewood theorem, in *Mathematische Zeitschrift* 32, pages 319–320 [37, 35]. The property holds for monomials, hence for polynomials, hence by approximation for step functions, using positivity throughout. In Borwein's words, "Karamata simplified Hardy and Littlewood's proof" [35]. (Check: 1914 to 1930 is 16 years.)

In 1932 Norbert Wiener proved his general Tauberian theorem, in the *Annals of Mathematics* 33, pages 1–100 [38, 39, 24]. The translates of a function f in $L^1(\mathbb{R})$ span a dense subspace exactly when its Fourier transform has no real zeros; the

Encyclopedia of Mathematics gives an equivalent form in terms of convolutions [39]. The theorem generalises the line from Tauber through Littlewood to Hardy and Littlewood, but no source says the proof uses those theorems.

In 1980 Donald Newman gave a short proof of the prime number theorem, which Don Zagier set out in 1997 [40, 41]. Its core is an Analytic Theorem: if f is bounded and its Laplace transform extends holomorphically to Re z ≥ 0, then the integral of f equals the value g(0) of that extension. The proof applies Cauchy's integral theorem on a contour that hugs the line Re z = 0. (Check: π(10^6) = 78 498, and π(x)/(x/ln x) is 1.1320 at 10^4 and 1.0845 at 10^6, slowly falling towards 1.) This is Tauberian theory in a very simple version. Newman's method yields a weaker form of the Wiener–Ikehara theorem, which is enough for the prime number theorem, and he proves that weaker form directly rather than using Wiener's [42]. Wiener is background, not a proof step.

In 2025 Lillian Pierce, Caroline Turnage-Butterbaugh and Asif Zaman posted a guide to Tauberian theorems for Dirichlet series with non-negative coefficients [43]. It covers theorems with and without an explicit remainder, and gives counterexamples showing that neither kind can be strengthened much without stronger hypotheses. It appeared in *Essential Number Theory* 5(2), pages 271–374, on 14 August 2026, under MSC 11M45 and 40E05 [44]. (Check: from arXiv submission on 22 April 2025 to publication took 479 days, about 1.3 years.) The guide discusses the Ikehara–Wiener theorem in its section 2.2.1 and its extension to poles of higher order [43]. It does not cite Newman, so there is no stated line from Newman to it.

Notes

1. Euler's Basel solution is dated 1734 (when he found it, a date only Wikipedia gives), 1735 (when it was presented, according to Monks and the Euler Archive) and 1740 (when it was printed). These are stages of one result.
2. Who first saw that rearrangement changes a sum is disputed. Castejón, Corbacho and Tarieladze (2021) say Dirichlet in 1827; Wikipedia credits Cauchy in 1833; Riemann's own text credits Dirichlet, citing Crelle IV (1829). The 1827 date has not been seen in a primary source.
3. Riemann's paper is dated 1867 by Dedekind's note of July 1867, and 1868 by volume 13 of the *Abhandlungen*, according to Wikipedia.
4. Toeplitz's paper (*Prace Matematyczno-Fizyczne* 22) is dated 1911 by Wikipedia and the *Dictionary of Scientific Biography*, but 1913 by Dym and Katsnelson (arXiv 0706.1868). The date 1911 is kept here.
5. Jacob Bernoulli published a proof that the harmonic series diverges in 1689 and credited his brother Johann, but Mengoli (1650) and Oresme (c. 1350, per Stammbach 1999) came earlier. The so-called Leibniz series was found by Mādhava's school, by Gregory (1671) and by Leibniz (1673), each independently.
6. Only Wikipedia's article on the harmonic series describes Mengoli's method of grouping in threes; MacTutor states the result but not the method.
7. That Hardy suggested to Littlewood the case he proved in 1911 rests on Wikipedia alone; neither the Encyclopedia of Mathematics nor Borwein mentions it.
8. The 1914 Hardy–Littlewood theorem is stated for non-negative coefficients by Wikipedia and the Encyclopedia of Mathematics, and with a one-sided condition by Borwein. The two statements are compatible.
9. Kronecker's lemma, Hölder means, Euler summation, Watson's theorem,

Ramanujan summation and statistical convergence were not traced, and no other 2025–26 result in MSC 40 was found.

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Approximation theory

Approximation theory asks how well simple functions, such as polynomials or rational functions, can stand in for complicated ones, and how fast the error shrinks as the simple functions grow. The chain starts with Chebyshev's best-approximation theorem of 1854 and runs in three lines. The first is density: Weierstrass in 1885, then Bernstein, Müntz, Stone and Mergelyan. The second is rates: Jackson and Bernstein, then Newman in 1964 and Stahl in 1992–93 for the rational approximation of |x|. The third is superposition: Kolmogorov and Arnold in 1956–57, then the universal approximation theorems for neural networks of 1989.

The newest step, Kolmogorov–Arnold Networks of 2024–25, is an engineering use of the superposition theorem, not a new theorem in this field. Chebyshev and Weierstrass stand side by side: no source says Weierstrass's proof uses Chebyshev's theorem, so the two results are independent.

Best approximation and density: Chebyshev to Weierstrass

In 1854 Pafnuty Chebyshev proved the criterion for best uniform approximation [1, 2]. A polynomial P of degree n is the best uniform approximation to a continuous function exactly when the error alternates in sign, reaching its maximum size, at n + 2 points. Chebyshev first stated it for rational functions, and the Chebyshev polynomials first appear in the same 1854 work [2, 1]. In modern form the proof runs as follows [3]. A better polynomial would differ from P with n + 1 sign changes, hence n + 1 roots, which a polynomial of degree n cannot have unless it is zero; with fewer alternations, a correcting polynomial lowers the maximum. Chebyshev's own argument has not been seen (note 1).

(Two quick checks. Among monic polynomials of degree n on [−1, 1], the smallest maximum is 2^(1−n): 0.0625 for n = 5 and 0.0078125 for n = 8. For e^x on [0, 1] the best straight line has slope e − 1 = 1.7182818285, and its error is ±0.1059334, reached with alternating signs at 0, at ln(e − 1) = 0.5413249 and at 1.)

In 1885 Karl Weierstrass proved, in a paper for the Berlin Academy, that every continuous function on a closed interval [a, b] is a uniform limit of polynomials [4, 5]. His method was convolution with the Gauss or heat kernel, the operation now called the Weierstrass transform [5, 6]. The Encyclopedia of Mathematics states the theorem but not the method (note 2). This theorem is the root of the density line: nearly every later result in it generalises or reproves it.

Rates and new proofs: Jackson, Bernstein and Müntz

In 1911 Dunham Jackson proved the first direct theorem on rates, in his Göttingen thesis written under Edmund Landau [8, 9]. If f has a bounded r-th derivative, its best error E_n(f) by polynomials of degree n is at most C(r)/n^r [7]. In terms of the modulus of continuity ω the estimate reads E_n(f) ≤ C·ω(f; 1/n) [4, 10]. The tool was convolution with the kernel now named after him, which the Encyclopedia of Mathematics says

was “first employed by D. Jackson” for this estimate [11]. The result strengthens Weierstrass’s theorem by adding a rate.

In 1912 Sergei Bernstein gave a constructive proof of the Weierstrass theorem, using what are now called Bernstein polynomials (note 3) [12, 13]. The polynomial B_n f takes the values f(k/n) and weights them with the binomial probabilities; the weak law of large numbers then shows that B_n f converges uniformly to f. (Check: for f(x) = |x − ½|, the scaled error √n·(B_n f − f)(½) is 0.389108, 0.397946 and 0.398843 for n = 10, 100 and 1000, tending to 1/√(2π) = 0.398942.) This is a new proof of Weierstrass’s theorem.

In the same year Bernstein won the Belgian Academy prize with a memoir answering a question of Charles de la Vallée Poussin [12, 14]. It contains his work on the best approximation of |x|, his inverse theorems, which show that a fast rate of approximation forces smoothness, and Bernstein’s inequality for the derivative of a polynomial [14]. In *Acta Mathematica* 37 he showed that the limit of 2n·E_2n(|x|) exists, and bounded it between 0.278 and 0.286 [14, 29]. (Check: computing n·E_n(|x|) on a 60 000-point grid gives 0.270484, 0.277518, 0.279489, 0.279998 and 0.280127 for n = 4, 8, 16, 32 and 64. The values rise towards Bernstein’s constant β = 0.2801695 [30, 31] and stay below 1/(2√π) = 0.282095. Grid values are lower bounds, not a proof.)

In 1914 Herman Müntz proved that the powers x^λ₁, x^λ₂, … span a dense set in C[a, b], with 0 < a < b, exactly when Σ 1/λ_k = ∞ [15, 17]. He measured the L^2 distance from x^λ to the span of the other powers through Gram determinants, which he evaluated as Cauchy determinants; an arXiv study says this was Müntz’s original method [18]. (Check: Σ $1/k^2$ up to 10^6 is 1.64493307 and stays finite, so the squares {x^(k^2)} are not complete; Σ 1/k up to 10^6 is 14.3927 and diverges.) The paper’s title, “Über den Approximationssatz von Weierstrass”, shows that it generalises Weierstrass’s theorem [16]. It answered a question Bernstein had posed in 1912, and Otto Szász extended it in 1916 in *Mathematische Annalen* 77 [16, 17].

Generalisations: Stone and Mergelyan

Marshall Stone generalised Weierstrass’s theorem in 1937 and gave a simplified version in 1948 [5, 19] (note 4). A subalgebra of the continuous functions C(K) on a compact space K that separates points and contains the constants is dense. The proof uses lattice closure: the algebra is closed under max and min, which follows from approximating |t| by polynomials (note 2).

In 1951 Sergei Mergelyan proved the complex counterpart (note 5) [20, 21, 22]. On a compact set K whose complement is connected, every function continuous on K and holomorphic in its interior is a uniform limit of polynomials. The sources do not describe his method. The result completes earlier work on the same problem by Joseph Walsh, by Mikhail Lavrentiev (1936, for sets without interior points) and by Mstislav Keldysh (1945, for the closure of a domain), which the Encyclopedia of Mathematics calls special cases and forerunners [20, 22]. No source says that Mergelyan’s proof uses their results.

Superposition and rational approximation

In 1956–57 Andrei Kolmogorov and Vladimir Arnold proved that every continuous function on the cube [0, 1]^n is a finite sum of compositions of continuous functions of one variable [23, 24, 25] (note 6). Arnold reduced three variables to two. Kolmogorov then went down to one variable, with fixed monotone inner functions that do not depend on the function represented. The theorem settles the continuous form of Hilbert’s thirteenth problem of 1900.

In 1964 Donald Newman showed that rational functions approximate |x| far better than polynomials do [26, 27, 28]. The best error E_n,n(|x|) by rational functions of degree n is at most 3e^(−√n) for n ≥ 4, in the *Michigan Mathematical Journal* 11, pages 11–14. He built an explicit rational function from the product ∏(x + a^k) with a = e^(−1/√n). (Check: the maximum errors of Newman’s function are 2.782 × 10^−2, 5.739 × 10^−3, 1.471 × 10^−3 and 4.175 × 10^−4 for n = 4, 9, 16 and 25, all well below 3e^(−√n).) Polynomials, by contrast, only reach an error of about β/n.

Bernstein had conjectured that his limit equals 1/(2√π), the near midpoint of his bounds. In 1985 Richard Varga and Amos Carpenter disproved this in *Constructive Approximation* 1, pages 333–348 [32]. They gave rigorous upper and lower bounds for β and used Richardson extrapolation to find it to about 50 digits [32, 27, 31]. (Check: β = 0.2801695, against 1/(2√π) = 0.2820948.) A version was received by *Matematicheskii Sbornik* on 27 March 1985 and printed in English in *Math. USSR-Sbornik* 57 (1987), pages 547–560 [29] (note 7). Their result is a counterexample to Bernstein’s conjecture.

In 1989 George Cybenko, and independently Kurt Hornik, Maxwell Stinchcombe and Halbert White, proved that neural networks with one hidden layer of sigmoid units are dense in C([0, 1]^n) [33, 34, 35]. Cybenko argued by duality, with the Hahn–Banach theorem and the Riesz representation theorem. Hornik, Stinchcombe and White wrote “We apply the Stone-Weierstrass Theorem” [35]; their proof rests on Stone’s theorem. They cite Kolmogorov’s 1957 theorem as background only, not in the proof (note 8).

In 1992–93 Herbert Stahl found the exact asymptotics of rational approximation to |x|: e^(π√n)·E_n,n(|x|) tends to 8 [26, 28]. This proved a conjecture of Varga, Ruttan and Carpenter, published in *Math. USSR-Sbornik* 74 [26]. He used potential theory and the study of extremal rational functions. In 2003 he extended the result to x^α in *Acta Mathematica* 190 [27]. (Check: 8e^(−π√25) = 1.206 × 10^−6 at n = 25, against 4.175 × 10^−4 for Newman’s function.) Stahl’s work rests on Newman’s 1964 bound and Vyacheslavov’s 1975 refinement, and on that conjecture [26, 27].

In 2024 Ziming Liu and co-authors proposed Kolmogorov–Arnold Networks, in which the edges of the network carry learnable functions of one variable [36, 23]. The paper was presented as a poster at ICLR 2025 [37]. It proves no new theorem: it is an architecture inspired by the representation theorem, which influenced it without being a proof step.

Notes

1. Chebyshev’s own 1854 argument has not been seen; the proof idea given is the modern one from Wikipedia’s article on the equioscillation theorem.
2. Weierstrass’s heat-kernel method and the lattice argument in Stone’s proof are described by Wikipedia; the Encyclopedia of Mathematics and MacTutor confirm

the theorems but not the methods.

3. Bernstein's proof of the Weierstrass theorem is dated 1911 by MacTutor and 1912 by the Encyclopedia of Mathematics, Wikipedia and the History of Approximation Theory site. The primary citation (Comm. Kharkov Math. Soc. 13) favours 1912, though the bound volume XIII carries 1913. His limit for |x| is dated 1913 by Lubinsky and 1914 (*Acta Math.* 37) by Wikipedia, the History of Approximation Theory site and Stahl (2003). The prize memoir with the inverse theorems is dated 1912; MacTutor dates the prize 1911. Jackson's thesis is dated 12 July 1911 (the defence, MacTutor) and 14 June 1911 (History of Approximation Theory); the year is the same.
4. MacTutor places the Stone–Weierstrass theorem in the early 1930s, while Wikipedia gives 1937 and 1948, probably discovery against publication.
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6. MacTutor credits Kolmogorov alone (1957) and dates Arnold's thesis on the thirteenth problem to 1961, while Wikipedia gives Kolmogorov 1956 and Arnold 1957.
7. The year of the Russian original in *Matematicheskii Sbornik* has not been confirmed; Math-Net.Ru gives the receipt date and the English printing of 1987. The first publication is the 1985 paper in *Constructive Approximation*.
8. Wikipedia dates the Hornik–Stinchcombe–White paper to July 1989; the paper itself gives *Neural Networks* 2, 359–366, received 16 September 1988 and accepted 9 March 1989.
9. Stahl's |x| theorem is *Mat. Sb.* 1992 and *Sb. Math.* 1993; *Acta Math.* 190 (2003) is the extension to x^α.
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Abstract harmonic analysis

Abstract harmonic analysis carries Fourier analysis from the line and the circle to groups. Its roots, Fourier series, Wiener's Tauberian theorem and Plancherel's theorem on the real line, belong to the Analysis chain. This field starts when the analysis moves to groups: with Hurwitz's invariant integral of 1897, the Peter–Weyl theorem of 1927 and Haar's invariant measure of 1933.

From Haar run von Neumann's and Pontryagin's cases of Hilbert's fifth problem, duality for all locally compact abelian groups, and the Plancherel theorem. Then the chain splits into two branches: Tate's adelic thesis of 1950, and Kazhdan's property (T) of 1967, which leads to computer-certified proofs for the automorphism groups Aut(F_n) of free groups in 2019 and 2021. No later link was found, so in 2026 the chain stops there. The Peter–Weyl theorem stands at the head of the representation line, but no source says a later proof uses it as a step.

Invariant integration: Hurwitz, Peter–Weyl and Haar

In 1897 Adolf Hurwitz built an invariant integral on Lie groups, in a paper for the Göttingen Academy [1, 2, 3]. He averaged over the group with a volume form that does not change under the group action. In 1900 David Hilbert posed his fifth problem: is every locally Euclidean group a Lie group, without assuming differentiability [4, 5]? It was a question, not a result, but it shaped the next half-century of this chain.

In 1927 Fritz Peter and Hermann Weyl proved that for a compact group G the matrix coefficients of the irreducible unitary representations form an orthonormal basis of $L^2(G)$ [6, 7], in *Mathematische Annalen* 97. The sources do not describe their method (note 1). (Check: for the symmetric group S_3 the irreducible representations have dimensions 1, 1 and 2, and $1^2 + 1^2 + 2^2 = 6 = |S_3|$, as the theorem requires for a finite group.) The sources call the theorem the beginning of the representation line, a historical statement rather than a proof link.

In 1933 Alfred Haar proved that every second-countable locally compact group carries an invariant measure [1, 3, 8]. He found it in 1932 and published it in the *Annals of Mathematics* in 1933 [8, 9]. The *Dictionary of Scientific Biography* calls his method "a bold direct approach" [9]. (Check: on the circle $\mathbb{R}/\mathbb{Z}$ the mean of t is $\int_0^1 t\,dt = ½$, and it is still ½ after a shift by 0.7, as an invariant measure demands.) Hurwitz's integral is a special case: Haar's introduction cites Hurwitz as a predecessor, and Haar's result covers all such groups, not only Lie groups [3, 1]. This is a generalisation of Hurwitz, not a use of his proof.

Hilbert's fifth problem and duality

Also in 1933 John von Neumann settled the compact case of Hilbert's fifth problem: every compact locally Euclidean group is a Lie group [4, 10, 11]. The paper was submitted on the same day as Haar's and printed next to it in *Annals* 34 [10] (note 2). It uses Haar's theorem; the *Dictionary of Scientific Biography* calls the result an

"immediate consequence" [4, 12]. According to the same source, von Neumann had found the basic idea six years earlier: closed subgroups of the general linear group are Lie groups [12]. How the proof passes from Haar measure to a Lie structure is not described in the sources (note 3).

In 1934 Lev Pontryagin proved his duality theorem for second-countable compact or discrete abelian groups, using the theory of characters [13]. With the same theory he solved the abelian case of Hilbert's fifth problem in 1934 [5, 13]. (Check: the group $\mathbb{Z}/2 \times \mathbb{Z}/4$ has exactly 8 characters, as many as its elements, and they are orthogonal up to a rounding error of 3.0 × 10^−16.) MacTutor and the *Dictionary of Scientific Biography* say Pontryagin applied Haar measure [8, 9]; Wikipedia says duality was combined with Haar measure only later (note 4).

Egbert van Kampen in 1935 and André Weil in 1940 extended duality to all locally compact abelian groups, by extending the theory of characters [13, 14]. This builds on Pontryagin's 1934 theorem; Universalis credits Pontryagin and van Kampen together with duality for these groups, around 1935 [15] (note 5).

In 1940 Weil and Henri Cartan proved that Haar measure exists and is unique on every locally compact group, not only the second-countable ones [1, 8]. Weil used the axiom of choice; Cartan did without it. (Check: from Haar in 1933 to Weil in 1940 is 7 years.) Both results generalise Haar's theorem.

Plancherel, Banach algebras and enough representations

In 1941 Mark Krein proved the Plancherel theorem for commutative groups: the Fourier transform extends to a unitary map from $L^2(G)$ onto $L^2(\hat{G})$, where $\hat{G}$ is the dual group [6, 13, 16]. The Encyclopedia of Mathematics and Vainerman cite his note "On a generalization of the Plancherel theorem to the case of Fourier integrals on a commutative topological group", *Doklady Akademii Nauk SSSR* 30(6), pages 482–486 [6, 16]. Wikipedia states the unitary extension without naming Krein [13]. (Check: on $\mathbb{Z}/12$ with x = (0, 1, …, 11), both $\|x\|^2$ and the squared norm of its unitary discrete Fourier transform are 506.) The result uses Haar measure and Pontryagin duality, since both are needed to state it; neither source says what the proof itself uses.

In the same year Israel Gelfand gave a new proof of Wiener's Tauberian theorem of 1932, which belongs to the Analysis chain, using his theory of Banach algebras [17, 18, 19]. It appeared in his 1941 paper in *Matematicheskii Sbornik*. The result rests on Wiener's: it proves the same theorem by a new method.

In 1943 Gelfand and Dmitrii Raikov proved that every locally compact group has enough irreducible unitary representations to separate its points [6, 20, 21]. It appeared in *Matematicheskii Sbornik* 13 [6]. The proof works with positive-definite functions and the Krein–Milman theorem. It fixes a Haar measure and works in the group algebra $L^1(G)$, so it uses Haar's theorem. Only Yoshizawa (1949) describes this argument, which he takes from Segal and Godement rather than from the 1943 original [21].

Two branches: Tate's thesis and property (T)

In 1950 John Tate proved the functional equation of Hecke L-functions in his Princeton thesis, which was published only in 1967 [22, 23, 24]. (Check: 1967 −

1950 = 17 years.) He integrated over the idele group with an invariant measure and applied Poisson summation [22]. Tate proved that the adeles are self-dual and derived an adelic Poisson summation formula [24]. This builds on duality and invariant integration on locally compact abelian groups, though neither source names Pontryagin's 1934 or Weil's 1940 work as used; Milne cites Weil only for the topology on the ideles [24]. Kenkichi Iwasawa found the same method independently, announcing it at the 1950 International Congress and describing it in a 1952 letter to Dieudonné [25, 26].

In 1952 Andrew Gleason, and Deane Montgomery with Leo Zippin, proved that every locally Euclidean group is a Lie group [27, 4, 28]. This settles Hilbert's fifth problem, proving the conjecture in full, and crosses into the theory of topological groups (MSC 22).

In 1959 Paul Malliavin proved that spectral synthesis fails on every non-discrete locally compact abelian group [6, 29]. He constructed a closed set that is not a set of synthesis, a counterexample to the spectral-synthesis question. A 2025 arXiv paper cites his three papers of 1959 [29]. This link stands apart from the rest of the chain.

From 1965 to 1976 Harish-Chandra found the Plancherel measure for semisimple Lie groups and the discrete series [30, 31, 32]. His tool was the characters of representations, treated as invariant eigendistributions, as the title of his 1965 *Acta* paper says [31]. The discrete series appeared in 1965–66 [31]; Langlands says he had proved their existence by 1964 [32]. The full Plancherel formula appeared in 1976 in the *Annals of Mathematics* 104, pages 117–201, according to Delorme's references (note 6) [33]. This work has no stated link to the rest of the chain.

In 1967 David Kazhdan introduced property (T): the trivial representation is isolated in the unitary dual of the group [34, 35, 36]. It appeared in *Functional Analysis and its Applications* 1(1) [35]. He proved that countable groups with property (T) are finitely generated [34]. In 1973 Grigory Margulis used it to build the first explicit expanders [34, 36, 37], in *Problemy Peredachi Informatsii* 9(4) [37]. The spectral gap that property (T) gives for SL(n, ℤ), n ≥ 3, yields a uniform expansion constant. This rests directly on Kazhdan's result that SL(n, ℤ) has property (T) for n ≥ 3.

In 2019 Marek Kaluba, Piotr Nowak and Narutaka Ozawa proved that $\mathrm{Aut}(F_5)$, the automorphism group of the free group on five generators, has property (T) [38, 39, 34]. They used Ozawa's characterisation of the property and a computer-certified sum-of-squares decomposition. The paper appeared in *Mathematische Annalen* (note 7). In 2021 Kaluba, Dawid Kielak and Nowak proved the same for every n ≥ 6, which together with the 2019 result covers all n ≥ 5 [40, 41]. They also gave explicit lower bounds for the Kazhdan constants of SL_n(ℤ), n ≥ 3, improving the known bounds for n > 6, in the *Annals of Mathematics* 193(2), pages 539–562 [40, 41]. The proof is computer-assisted, by Ozawa's sum-of-squares method [34], and builds on the 2019 result. Both rest on Kazhdan's property (T).

Notes

1. Peter–Weyl in 1927 covers compact Lie groups according to arXiv 2201.11144, which matches the paper's title ("closed continuous group") in the Encyclopedia of Mathematics. Wikipedia and the Encyclopedia of Mathematics' Peter–Weyl entry state it for compact topological groups, which would need Haar measure,

published only in 1933. The method of the original proof is not described in the sources.

2. Von Neumann’s compact case is dated 1933. Rosenberg says it was submitted the same day as Haar’s paper and printed next to it in *Annals* 34, so MacTutor’s date of 1929 does not hold.
3. That von Neumann combined Haar measure with Peter–Weyl theory to obtain a faithful finite-dimensional representation is not stated in the sources; Rosenberg links Peter–Weyl only to a later corollary.
4. MacTutor and the *Dictionary of Scientific Biography* (both on the MacTutor site) say Pontryagin applied Haar measure, while Wikipedia says duality was only later combined with it. Neither dates the use.
5. Wikipedia credits van Kampen in 1935 and Weil in 1940; the Encyclopedia of Mathematics cites van Kampen’s note of 1934; Universalis dates the duality to about 1935.
6. Only Delorme’s reference list gives 1976 for Harish-Chandra’s Plancherel formula; MacTutor and Langlands give no year.
7. The 2019 Kaluba–Nowak–Ozawa paper appeared in *Mathematische Annalen*, not in the *Annals*; the *Annals* 193 paper is Kaluba–Kielak–Nowak.

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Integral transforms

An integral transform turns a function into another function by integrating it against a fixed kernel, so that a hard problem, such as a differential equation, becomes an easier one. The chain starts with the integrals depending on a parameter that Euler used for differential equations, and with Laplace, who transformed whole equations between 1782 and 1812.

From there it runs in three lines. The first is transforms and their inversion: Mellin in 1896–97 and Bernstein in 1928. The second is operational calculus: Heaviside's formal rules, made rigorous by Bromwich around 1916 and algebraic by Mikusiński, who built on Titchmarsh's convolution theorem. The third is the Radon transform of 1917 and its use in tomography. Cormack did not build on Radon: he rediscovered the inversion on his own and learned of Radon's work only in the early 1970s. No 2025–26 result in this field could be confirmed (note 7); the Fourier line belongs to the Analysis chain.

Laplace and Mellin: from Euler's integrals to inversion on a line

Leonhard Euler invented integral transforms while studying second-order differential equations [2]. Between 1744 and 1769 he sought solutions in the form of definite integrals depending on a parameter, such as ∫ e^(ax) X(x) dx, and the gamma function belongs to the same circle of ideas [1]. Deakin places the decisive use in a fragment of 1763 and in the *Institutiones Calculi Integralis* of 1769 [2].

Pierre-Simon Laplace went one step further in 1782–85 [1, 3, 4]. He applied the transform to a whole difference equation, solved the transformed equation and transformed the solution back (note 1). His generating functions appear in the *Théorie analytique des probabilités* of 1812 (note 2) [3, 4]. (Check: the rule L[t^n] = n!/s^(n+1) gives 0.375 at n = 3, s = 2, and 10.534979423868 at n = 5, s = 1.5, matching the integrals computed directly.) Laplace built on Euler's integrals as an influence, though no source states it outside Wikipedia (note 1).

In 1859 Bernhard Riemann used what is now called the Mellin integral for the zeta function [5, 6]. His paper already writes Π(s − 1)ζ(s) = ∫₀^∞ $x^{(s-1)/(e}$x − 1) dx, where Π(s − 1) is the gamma function Γ(s), and inverts it along a vertical line [6].

In 1896–97 Hjalmar Mellin introduced the transform that bears his name, together with its inversion formula on a vertical line (note 3) [9, 10]. He used it for the gamma function, hypergeometric functions and Dirichlet series. Shifting the contour and collecting residues gives asymptotic expansions. (Check: the Mellin transform of e^(−x) at s = ½ is 1.772453850906, equal to Γ(½) = √π.) Riemann has priority for the integral and its inversion [6], though no source says Mellin built on him.

Operational calculus: Heaviside, Bromwich, Titchmarsh and Mikusiński

From the 1880s to 1893 Oliver Heaviside treated the operator p = d/dt in circuit equations as if it were an ordinary algebraic quantity [7, 8]. MacTutor dates this to 1880–1887, when he replaced d/dx by p [7] (note 4). He gave formal rules and no proofs. According to MacTutor, a paper he sent to the *Proceedings of the Royal Society* was rejected by Burnside as referee [7]. (Check: the transform pair $1/\sqrt{p} \mapsto 1/\sqrt{(\pi t)}$, inverted numerically by Talbot's method, gives $0.564189583548 = 1/\sqrt{\pi}$ at t = 1; the sources do not say Heaviside used this pair.)

Thomas Bromwich gave the first rigorous justification of Heaviside's calculus [7, 8, 11]. He wrote the operators as the contour integral $(1/2\pi i)\int F(p)e^{(pt)}\,dp$ along a vertical line. (Check: inverting 1/(s + 1) this way gives $0.367879441171 = e^{(-1)}$ at t = 1.) His series of papers began in 1916, according to the *Dictionary of Scientific Biography*, and the main paper appeared in the *Proceedings of the London Mathematical Society* (2) 15, pages 401–448, in January 1917 [11, 12] (note 5). His work repaired the gap in Heaviside's calculus. It also uses the Laplace transform: Wikipedia lists him in its history and Deakin's history cites him, though the Encyclopedia of Mathematics names no one [1, 19].

In 1926 Edward Charles Titchmarsh proved that Laplace convolution on $[0, \infty)$ has no zero divisors [15, 16]. More precisely, the infimum of the support of $f*g$ is the sum of the infima of the supports of f and g. He proved it by complex-variable methods on the zeros of integral functions, using results of Phragmén–Lindelöf, Jensen, Carleman and Valiron [15]. The paper is "The zeros of certain integral functions", *Proc. LMS* (2) 25, pages 283–302 [15, 16] (note 4). (Check: convolving the indicators of [1, 2] and [0.5, 1] gives a function whose support starts at 1.5000.) Jan Mikusiński gave a new proof in 1953, in *Studia Mathematica* 13, pages 56–58 [15, 16].

Mikusiński turned operational calculus into algebra in 1944 and 1953 [20, 21] (note 4). He defined operators as elements of the field of quotients of the convolution ring of continuous functions on $[0, \infty)$. This gave an algebraic justification of Heaviside's calculus. It rests on Titchmarsh's theorem of 1926: because the ring has no zero divisors, its field of quotients exists. The problem itself was posed by Heaviside's rules.

Positive measures and the standard textbook

In 1928 Sergei Bernstein proved that the completely monotone functions are exactly the Laplace transforms of positive measures [17, 18] (note 4). A function is completely monotone if its derivatives alternate in sign: $(-1)^k f^{(k)} \geq 0$ for every k. (Check: $1/(1 + x) = \int e^{(-xt)} e^{(-t)}\,dt$ is 0.5 at x = 1, and $(-1)^k f^{(k)}(2) > 0$ for k = 0 to 5.) The definition is built on the Laplace transform.

In 1937 Gustav Doetsch published *Theorie und Anwendung der Laplace-Transformation* (Springer, Berlin) [1, 19]. The book made the Laplace transform the standard tool, with a systematic theory and its applications. Deakin says it gave the transform "its modern shape" [19].

The Radon transform and tomography

In 1917 Johann Radon introduced the transform now named after him, with an inversion formula: the integrals of a function over all lines in the plane determine the function [13, 14]. MacTutor describes it as recovering functions from their integrals over manifolds [13]. The inversion in its modern form is a back-projection, the average over all lines through a point, applied after a ramp filter [14]. Radon's own 1917 derivation is not described in the sources. (Check: for the indicator of the unit disk, the integral along a line at distance $p = 0.5$ from the centre is $2\sqrt{(1 - p^2)} = 1.7320508076$.)

In 1963 and 1964 Allan Cormack showed how to reconstruct an object from line integrals of X-ray absorption [22, 23]. He expanded both the attenuation function and the data in Fourier series, which led to integral equations of Abel type [22]. The papers appeared in the *Journal of Applied Physics*, under the title "Representation of a function by its line integrals, with some radiological applications" [23]. This was a rediscovery, not a use of Radon: Cormack learned of Radon's work only in 1971 according to Cherry, or "until the early seventies" in his own Nobel lecture [22, 23].

In 1979 Cormack and Godfrey Hounsfield shared the Nobel Prize in Physiology or Medicine "for the development of computer assisted tomography" [24, 22, 23]. The prize rests on Cormack's 1963–64 work. (Check: from Radon in 1917 to Cormack in 1963 is 46 years, and from 1963 to the prize is 16 years.)

Notes

1. Only Wikipedia states that Euler's integrals influenced Laplace and that Laplace in 1785 transformed a whole difference equation; Deakin's 1985 paper covers Euler but not Laplace.
2. Wikipedia dates Laplace's *Théorie analytique* to 1814, the second edition; MacTutor and the Encyclopedia of Mathematics agree on 1812 for the first.
3. Mellin's paper is dated 1896 by the Encyclopedia of Mathematics and 1897 by Wikipedia.
4. Other dates differ between sources. Heaviside's calculus is dated 1880–1887 by MacTutor, 1892 by the Encyclopedia of Mathematics and 1893 by Wikipedia. Titchmarsh's theorem is 1926 in Wikipedia and the AMS paper by Doss (*Proc. LMS* (2) 25), but 1924 in the Encyclopedia of Mathematics. Bernstein's *Acta Mathematica* 52 paper is 1928 in the Encyclopedia of Mathematics and in Wikipedia's text, but 1929 in Wikipedia's references. Wikipedia places Mikusiński in the 1930s, while MacTutor and the Encyclopedia of Mathematics give 1944 and 1953.
5. Bromwich's series began in 1916 (*Dictionary of Scientific Biography*); the paper Wikipedia cites is in a *Proc. LMS* issue dated January 1917; Deakin cites a *Philosophical Magazine* paper of 1919.
6. The Hankel and Hilbert transforms, Post–Widder inversion, the Paley–Wiener theorems and Helgason's range theorem were not traced.
7. The only 2025–26 item found is a formalisation of Laplace inversion in the Lean 4 proof assistant (arXiv 2608.07384, August 2026), from one source and not refereed [25]. Ramm's claimed solutions of the Pompeiu problem are left out.

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Integral equations

An integral equation has the unknown function inside an integral, as in φ(x) = f(x) + λ∫K(x, t)φ(t) dt. The chain starts with Abel's inversion of the generalised tautochrone in 1823. It runs through successive approximation (Liouville, Neumann) and Volterra's equations, and turns on Fredholm's theory of determinants of 1900–1903.

From Fredholm three lines lead on. One runs through Hilbert, Schmidt, Riesz and Schauder to compact operators, where it passes to the Analysis chain. A second runs through the Wiener–Hopf factorisation of 1931. A third leads to Fredholm determinants as laws of random matrices: Tracy and Widom in 1992–94, computed by Bornemann in 2010. The latest status found is a survey of 2021: there is still no general constructive method for the Wiener–Hopf factorisation of matrices. Schmidt's theory of symmetric kernels was developed independently of Fredholm's.

Abel, Liouville and the method of successive approximation

In 1823 Niels Henrik Abel solved the first integral equation, the generalised tautochrone problem, in Hansteen's Norwegian journal [1, 2, 3]. Given the time of descent as a function of height, he found the shape of the curve by inverting the integral. In modern terms the equation applies a fractional integral of order ½, which a fractional derivative undoes [3]. A German version, "Auflösung einer mechanischen Aufgabe", appeared in the first volume of Crelle's journal in 1826, from page 153 [4] (note 1). (Check: for f(x) = x the solution is φ(s) = 2√s/π, and ∫₀ˣ φ(s)/√(x − s) ds gives back 0.5, 1.0 and 2.0 at x = 0.5, 1 and 2.)

In 1837–38 Joseph Liouville rewrote second-order linear differential equations as integral equations by integrating twice, then solved them by iteration (note 2) [5, 6]. No source says he used Abel's work.

In 1877 Carl Neumann gave the first rigorous proof that successive approximation converges for equations of the second kind [6, 5, 7] (note 3). He wrote the resolvent as a power series in λ. The Encyclopedia of Mathematics notes that it converges "only for small values of" λ [7]. (Check: for φ(x) = 1 + λ∫₀¹ xt φ(t) dt the exact solution has φ(1) = 1 + (λ/2)/(1 − λ/3). This is 1.75 at λ = 1, where the iterates converge to it, and −2.75 at λ = 5, where 60 iterates have grown to 7.7 × 10^13.)

In 1896–97 Vito Volterra treated equations whose upper limit of integration is the variable x [8, 5, 9]. They have a unique continuous solution for every finite λ [5]. By 1897 he had a convergent iteration for kernels that vanish for y > x [9], and the Neumann series of such a kernel converges for all λ [7]. (Check: for φ(x) = 1 + λ∫₀ˣ φ the Neumann sum is e^(λx), which at x = 1 and λ = 10 is 22026.46579.)

In 1896 Henri Poincaré, studying the vibrating membrane, conjectured that the solution of such an equation is a meromorphic function of λ [5]. It was a conjecture, not a theorem (note 4).

Fredholm’s determinants and the road to compact operators

In 1900 and 1903 Ivar Fredholm built his determinant theory [8, 7, 9]. He defined a determinant D(λ) and its minors, proved that they converge for all λ, and showed that the equation has a unique solution whenever D(λ) ≠ 0. This yields the Fredholm alternative: either the equation is uniquely solvable for every right-hand side, or the homogeneous equation has a non-zero solution. D is the limit of n × n determinants built from the values K(x_i, y_j), by analogy with the matrix equation (I + F)U = V [9]. (Check: for K(x, t) = xt, the Nyström determinant of I − λK equals 1 − λ/3, that is 0.666666666667, 0.033333333333 and −0.666666666667 at λ = 1, 2.9 and 5.)

Fredholm’s theory answered a problem posed by Neumann’s iteration, which converged only under restrictions [7, 9]. It also proved Poincaré’s conjecture, since D(λ) is an entire function and the solution a quotient of entire functions [5]. MacTutor adds that Volterra’s work influenced Fredholm [8].

From 1904 to 1910 David Hilbert built the eigenvalue theory of symmetric kernels and of quadratic forms in infinitely many variables, the start of what became Hilbert space [10, 6]. He justified Fredholm’s passages to the limit, introduced complete continuity, and diagonalised the forms. The work was collected in his *Grundzüge* of 1912 (note 5). It rests directly on Fredholm’s 1903 theory.

From 1905 to 1908 Erhard Schmidt developed his own theory of symmetric kernels [11, 7]. It brought the Gram–Schmidt orthonormalisation, adjoint pairs of eigenfunctions for non-symmetric kernels (1907), and the geometry of ℓ^2, the space of square-summable sequences (1908). His method splits a kernel into a degenerate kernel plus a small remainder. He reproved Hilbert’s results of 1904–06 more simply, and worked independently of Fredholm (note 6).

Frigyes Riesz extended Fredholm’s theorems to every compact operator on a complete function space [7, 12, 13, 14]. He worked from finite-dimensional kernels and the closed ranges of I − K. He finished the paper on 19 January 1916, and the war delayed its printing in *Acta Mathematica* 41 (note 7) [14, 15]. It rests on Fredholm’s theory [7, 13] and on Hilbert’s notion of complete continuity [7, 14].

In 1921 Torsten Carleman extended Fredholm’s method to square-integrable kernels [7, 5]. Bornemann notes that Hilbert in 1904 and Carleman in 1921 introduced the regularised determinant $\det_2$ for Hilbert–Schmidt kernels [16]. The work rests on Fredholm’s.

In 1930 Juliusz Schauder introduced the adjoint operator in Banach spaces and proved the Riesz–Schauder theorems [7, 5, 14]. By duality, the Fredholm alternative holds on Banach spaces. Pokroky calls the Banach–Schauder adjoint of 1930 “the missing piece” of the theory of compact operators [14]. It rests on Riesz’s work.

Wiener–Hopf factorisation

In 1931 Norbert Wiener and Eberhard Hopf solved the convolution equation on a half-line, motivated by radiative equilibrium in stars [17, 18, 19]. The paper appeared in the *Sitzungsberichte* of the Berlin Academy, pages 696–706 [17]; MacTutor places the work in Hopf’s Harvard years, from 1930 [19]. They used an integral transform, the Laplace transform in the original, according to Kisil and co-authors [18]. The kernel’s transform is split into factors analytic in complementary half-planes, and an extended

Liouville argument finishes the proof [18, 17].

In 1958 Mark Krein treated half-line convolution equations in Banach and Hilbert function spaces, and with Israel Gohberg the matrix version for systems [17, 20, 21]. Both papers appeared in *Uspekhi Matematicheskikh Nauk* 13, numbers 2 and 5. The Encyclopedia of Mathematics places them within the factorisation theory that grew from Wiener and Hopf [17]. The Gohberg–Krein paper is also cited for the stability of the partial indices of a matrix factorisation [22].

In 2021 Anastasia Kisil, David Abrahams, Gennady Mishuris and Sergei Rogosin surveyed the field in the *Proceedings of the Royal Society A* 477 [18, 26]. According to the survey, there is no general method to determine the partial indices or to construct a general matrix Wiener–Hopf factorisation; it is done case by case (note 8).

Fredholm determinants in random matrix theory

In 1992–94 Craig Tracy and Harold Widom found the law of the largest eigenvalue of a large random Hermitian matrix (the Gaussian unitary ensemble) [23, 24]. It is the Fredholm determinant $F_2(s) = \det(I - K_{\mathrm{Airy}})$ on $L^2(s, \infty)$, where K_{Airy} is the Airy kernel. They expressed it through a Painlevé transcendent by identities for integrable operators. (Check, by Nyström discretisation: $F_2(-3) = 0.080319552939$, $F_2(-2) = 0.413224142505$ and $F_2(0) = 0.969372828355$, and the mean of the distribution is about -1.771087.) The definition is built on Fredholm’s determinant.

In 2008, in a paper printed in 2010, Folkmar Bornemann showed how to compute Fredholm determinants with exponential convergence for analytic kernels [16, 25]. He discretised with Nyström’s method on Gauss–Legendre nodes, and proved that the error tracks the quadrature error. (Check: $F_2(-2)$ agrees to within $10\text{^}{-14}$ for 40, 60 and 80 nodes.) His method applies directly to the Tracy–Widom laws.

Notes

1. Abel’s Norwegian paper of 1823 has priority over the German version in Crelle (1826). The end page of the Crelle paper has not been seen and is not given.
2. Liouville’s integral-equation work is dated 1837 in Carothers’ 1993 colloquium notes and 1838 in the Encyclopedia of Mathematics. The “Linköping notes” are Carothers’ text on a Linköping course page, so they count as one source.
3. Neumann’s series is dated 1877 by the Encyclopedia of Mathematics and Carothers’ notes, while the *Dictionary of Scientific Biography* entry on Fredholm dates his iteration to 1884.
4. Only the Encyclopedia of Mathematics states Poincaré’s conjecture and its proof by Fredholm. The *Dictionary of Scientific Biography* entry on Fredholm mentions Poincaré’s work of 1895–96 on $\Delta u + \lambda u = h$, but its text is garbled at that point. Only MacTutor states Volterra’s influence on Fredholm.
5. The Encyclopedia of Mathematics lists Hilbert under 1912 (the *Grundzüge*) but dates the work to 1904 in its text, a difference of book against papers.
6. Schmidt’s independence from Fredholm rests on two pages of the Encyclopedia of Mathematics; MacTutor does not mention Fredholm.
7. Springer’s metadata dates Riesz’s *Acta Math.* 41 paper to December 1916 (issue 1, pages 71–98), while MacTutor and the Encyclopedia of Mathematics give 1918.

Pokroky says he finished it on 19 January 1916 and the war delayed it.

8. The statement that no general factorisation method exists rests on the 2021 survey and was not rechecked against later literature. Mercer's theorem, Noether's index and Muskhelishvili's work were not traced, and no 2025–26 result was searched for.

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Operator theory

Operator theory studies linear maps on spaces of infinite dimension, above all their spectra and the subspaces they leave invariant. The chain starts with Cauchy's spectral theorem for symmetric matrices and with Fredholm's integral equations of 1900–1903. Hilbert and Riesz turned these into spectral theory and the theory of compact operators, leading up to the spectral theorem of von Neumann and Stone of 1929–32.

Two lines then branch. One is the invariant subspace problem: proved for compact, polynomially compact and Lomonosov operators, and refuted on Banach spaces by Enflo and Read. The other is a pair of famous problems: Kadison–Singer, solved in 2013, and Halmos's similarity problem, answered in the negative by Pisier in 1996. In 2026 the invariant subspace problem on separable Hilbert space is still open. Enflo's and Neville's proofs of 2023 are unrefereed preprints, and nothing accepted builds on them. Banach's textbook of 1932 also stands without a link into the later results.

From matrices to compact operators: Cauchy to Riesz

Augustin-Louis Cauchy proved the spectral theorem for real symmetric matrices in 1826 or 1829 (note 1) [1, 2, 3]. Such a matrix has real eigenvalues and can be diagonalised by an orthogonal change of basis. (Check: for a random symmetric 5 × 5 matrix the eigenvalues are real, and rebuilding the matrix as U diag(w) U^T from its eigenvectors U and eigenvalues w misses it by only 3.7 × 10^−15.)

In 1900 and 1903 Ivar Fredholm proved his alternative for integral equations: u − λKu = f is solvable for every f unless λ is an eigenvalue [4, 5]. He worked with determinant expansions of the kernel K. The idea of Hilbert space grew out of this work [4]. (Check: the discretised Volterra operator, integration from 0 to x on a grid of step h = 0.0025, has spectral radius 0.0025 = h, which tends to 0, while its norm is 0.6374, close to 2/π. An operator can thus have spectrum {0} and still be far from zero.)

From 1904 to 1906 David Hilbert built a complete eigenvalue theory for symmetric kernels, then for quadratic forms in infinitely many variables [4, 2, 6, 7]. In 1906 he added the continuous spectrum. His work rests directly on Fredholm's of 1903.

In 1916 Frigyes Riesz created the abstract theory of compact operators, with a German version in 1918 (note 2) [2, 8]. He derived the Fredholm alternative from compactness alone. Universalis says Riesz was the first to see that compactness by itself recovers Fredholm's theory [8], so the work builds on Fredholm's. That it also uses Hilbert's 1904 work is inferred from the content, not stated by a source. (Check: in a model of the unilateral shift, which moves each basis vector one place along, the kernel has dimension 0 and the cokernel dimension 1, so the Fredholm index is −1.)

The spectral theorem and its abstractions

From 1929 to 1932 John von Neumann and Marshall Stone proved the spectral theorem for unbounded self-adjoint and normal operators [2, 9, 10, 1]. They used

spectral measures and the Cayley transform, which turns a self-adjoint operator into a unitary one. The two found it at the same time, Stone partly independently of von Neumann. Mackey describes it as the extension of Hilbert's spectral theorem from bounded to unbounded operators [9], and Murray as a reformulation of Hilbert's theory for bounded symmetric operators [10]. It rests on Hilbert's work of 1904–06, not on Cauchy's.

In 1932 Stefan Banach published *Théorie des opérations linéaires*, the first textbook of operator theory, set in the framework of Banach spaces [2, 11]. A Polish version had appeared in 1931 [11]. The book belongs to the Analysis chain, and no later result here is said to use it.

In 1941 Israel Gelfand developed spectral theory in normed algebras, in his *Normierte Ringe*, and proved the formula r(T) = lim ‖Tⁿ‖^(1/n) for the spectral radius [2, 12, 13]. According to Harrell, this builds on the spectral theory of von Neumann and Stone [2] (note 3).

The invariant subspace problem

Does every bounded operator on a space of dimension above one have a closed invariant subspace other than 0 and the whole space? In 1954 Nachman Aronszajn and Kennan Smith gave the first published proof for compact operators [14, 15]. Wikipedia states it for any Banach space; the *Nieuw Archief* article states the Hilbert-space case. Von Neumann had an unpublished proof for Hilbert space, on which their result builds.

In 1966 Allen Bernstein and Abraham Robinson proved that polynomially compact operators have invariant subspaces [14, 15]. These are operators T for which p(T) is compact for some non-zero polynomial p. Their proof used non-standard analysis, approximating the operator by finite-dimensional ones. The *Nieuw Archief* article says it extended the Aronszajn–Smith theorem [15]; that the proof uses that theorem is not stated. In the same issue of the same journal, Paul Halmos proved the theorem again by standard analysis, translating their proof [14, 15].

In 1973 Victor Lomonosov proved that any operator commuting with a non-zero compact operator has a hyperinvariant subspace, one left invariant by every operator that commutes with it [14, 15]. His proof uses Schauder's fixed-point theorem. The *Nieuw Archief* article calls his technique entirely different from the earlier line, so it does not rest on Aronszajn–Smith or Bernstein–Robinson [15]. (Check: from 1954 to 1973 is 19 years.)

Per Enflo then showed that the problem fails on Banach spaces [14, 15]. He built, by a long explicit construction, a Banach-space operator with no invariant subspace. He announced it in 1975–76, at the Séminaire Maurey–Schwartz and in Toronto (note 4). The paper was submitted in 1981 and printed in *Acta Mathematica* in 1987. It is a counterexample to the Banach-space question.

In 1984 and 1985 Charles Read gave further counterexamples, among them the first on a classical space, ℓ^1 [15, 14]. Whether he built on Enflo's work or worked independently is disputed (note 5).

In 2011 Spiros Argyros and Richard Haydon built a Banach space on which every operator is a scalar plus a compact operator, in *Acta Mathematica* 206; the arXiv version is from 2009 [14, 22, 23, 24]. On this space every operator has an invariant

subspace. The paper notes, after Lindenstrauss, that this follows from "the main theorem of [10] or [26]" of its references [22]. A 2025 survey gives the reason: every compact operator has many invariant subspaces, so every operator of the form λI + K does too [25]. The result therefore uses the theorem on compact operators, in the line of Aronszajn–Smith and Lomonosov (note 6).

In May 2023 Enflo posted a claimed proof that every bounded operator on Hilbert space has a closed non-trivial invariant subspace [27, 14]. A second version followed in April 2024. As of 29 October 2026 it has no journal reference on arXiv and is unrefereed. Charles Neville posted an independent claim for separable Hilbert space on 17 July 2023, with a second version on 21 July 2023 [14, 28]. It too had no journal reference as of 29 October 2026. These are claims only, and nothing accepted builds on them.

Kadison–Singer and Halmos's similarity problem

In 1959 Richard Kadison and Isadore Singer posed their problem, in the *American Journal of Mathematics* 81 [16, 17, 18]. Does every pure state on the diagonal maximal abelian subalgebra of the operators on ℓ^2 extend uniquely to a pure state on all operators? It was a question, not a result.

In 1979 Joel Anderson reformulated it as the paving conjecture, which is equivalent to it, in the *Transactions of the AMS* 249 [16, 17]. In 2004 Nik Weaver gave another equivalent form, KS_2, a statement about discrepancy, in *Discrete Mathematics* 278 [16, 17]. Both rest on the Kadison–Singer problem as reformulations.

In 2013 Adam Marcus, Daniel Spielman and Nikhil Srivastava proved the Kadison–Singer conjecture through Weaver's KS_2 and Anderson's paving [17, 26, 18]. Their tools were interlacing families of polynomials and bounds on the roots of mixed characteristic polynomials. The proof appeared in the *Annals of Mathematics* 182 in 2015, and won the Pólya Prize in 2014 [26, 18]. (Check: from 1959 to 2013 is 54 years.)

In 1970 Halmos published "Ten problems in Hilbert space", a list of open questions that included the similarity problem [19, 20, 21]. Is every polynomially bounded operator similar to a contraction? In February 1996 Gilles Pisier answered no: he built a polynomially bounded operator that is not similar to a contraction [19, 21, 20]. It appeared in the *Journal of the AMS* 10 (1997), pages 351–369. (Check: von Neumann's inequality says $\|p(T)\| \le \max |p|$ on the unit circle for every contraction T. For a random 6×6 contraction and a random cubic p, $\|p(T)\| = 1.4483$ against $\max |p| = 3.0919$. Every operator similar to a contraction is therefore polynomially bounded; Pisier showed the converse fails.) The result answers Halmos's question by a counterexample.

Notes

1. Cauchy's spectral theorem is dated 1826 by Harrell and 1829 by Brechenmacher, who quotes the 1829 paper on the secular equation.
2. Riesz's dates of 1916 and 1918 are the Hungarian original and the German version (Harrell: the translation came "two years later"), not a real disagreement.
3. Only Harrell states that Gelfand built on the von Neumann–Stone theory.

4. Enflo's counterexample was announced in 1975–76. Wikipedia gives a proposal of 1975 and an outline of 1976; the *Nieuw Archief* article gives the Séminaire Maurey–Schwartz of 1975–76 and Toronto in 1976. Both give *Acta Math.* 1987.
5. The *Nieuw Archief* article says Read built his first counterexample "following the ideas of Enflo", but calls his 1985 ℓ^1 example "characteristically different from and simpler than Enflo's".
6. The references [10] and [26] cited by Argyros and Haydon have not been identified; Wikipedia names Aronszajn–Smith and Lomonosov.
7. The von Neumann inequality itself and the Sz.-Nagy dilation theorem were not traced, and no 2025–26 result in the field was confirmed.

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Calculus of variations and optimisation

The calculus of variations asks which curve, surface or function makes a quantity as small or as large as possible. Its chain starts with Zenodorus's isoperimetric comparisons (c. 200–140 BC). The subject proper was founded by Johann Bernoulli's brachistochrone challenge of 1696, which led to Euler's general method (1744), Lagrange's analytic δ-method (1755) and the name of the subject (1766).

The 19th century added Steiner's symmetrisation, with its gap about existence, and Weierstrass's objection to the Dirichlet principle, which Hilbert repaired. Hilbert's 19th problem (1900) was settled by De Giorgi and Nash (1956–58). Douglas solved Plateau's problem (1931) and Pontryagin opened optimal control (1956). The latest link is Milman and Neeman's proof of the double, triple and quadruple bubble conjectures, published in *Acta Mathematica* in 2025.

Several links stand apart. Galileo's wrong guess is refuted by the 1697 solution but not used by it. Zenodorus and Steiner, Steiner and Hurwitz, and the 2000 double bubble proof and Milman–Neeman are independent of each other. No source says that Pontryagin's principle was proved from a specific result of the classical calculus of variations, only that the two are connected; and the legend of Dido leads to no proof at all.

From isoperimetry to the brachistochrone

Zenodorus, working around 200–140 BC, showed that among polygons with equal perimeter and the same number of sides the regular one has the greatest area, and that the circle beats them all [1, 2, 3]. His work is lost; it is known through Theon and Pappus. A quick check with perimeter 1: the circle encloses $1/(4\pi) \approx 0.07958$, the square only $1/16 = 0.06250$. Zenodorus builds on Archimedes' theorem on the area of the circle; MacTutor names the theorem, and the *Dictionary of Scientific Biography* confirms that Zenodorus refers to Archimedes several times [1, 3]. The sources differ on whether he proved the polygon case or took it as obvious (note 6).

In 1638 Galileo guessed, wrongly, that a circular arc is the fastest path for a body sliding between two points under gravity [4, 5]. In 1696 Johann Bernoulli posed this brachistochrone problem as a challenge in *Acta Eruditorum* [4, 5].

In 1697 Newton, Leibniz, Johann Bernoulli and his brother Jakob found, in parallel, that the fastest curve is a cycloid [4, 5]. Johann used Fermat's principle of least time, treating the falling body like a light ray; Jakob worked with second differentials. Their answers rest on the problem Johann had posed, and they refute Galileo's arc. A quick check: from the origin to the point (1, 1) with $g = 9.81\ \text{m/s}^2$, the cycloid takes 0.58290 s and the straight line 0.63855 s. MacTutor counts Johann's 1697 solution, while Wikipedia calls his derivation flawed (note 6).

Euler, Lagrange and the name of the subject

In 1744 Euler published *Methodus inveniendi* at Lausanne and Geneva [5, 6]. It gave a general method, and the equation that extremals must satisfy, but kept Johann Bernoulli's geometrical approach. MacTutor says it generalised the problems of the Bernoulli brothers; Wikipedia says Euler refined Jakob's methods [4, 5]. The fully analytic form came with Lagrange in the 1750s (note 1). A small check of the idea: the discrete minimiser of the Dirichlet energy with 50 interior points is a straight line, to within a maximum deviation of 4.2×10^{-15}.

In a letter to Euler dated 12 August 1755, Lagrange introduced the analytic δ-method [7, 8, 9]. One varies the unknown function itself and removes the δy terms by integration by parts. His Turin memoir was written by 1759 and published in 1762 (note 5). The gap between the two works is $1755 - 1744 = 11$ years. The new method builds on the problem Euler had treated in 1744, but no source says Lagrange's derivation uses Euler's method; MacTutor and the DSB call it new and purely analytic [7, 9].

In 1766 Euler named the subject "calculus of variations", after Lagrange's variations [7, 8]. The name rests directly on Lagrange's 1755 method.

Existence, Dirichlet's principle and Hilbert's problem

In 1838 Steiner showed that if a shape of given perimeter with maximal area exists, it must be the circle [10, 11]. His tool was symmetrisation, which never increases the perimeter. Whether an optimum exists at all he left open. The date is 1838 in Wikipedia and in arXiv 2603.14663, but 1842 in Alama's handout (note 2). Steiner's argument is independent of Zenodorus: no source says it uses his work.

In 1870 Weierstrass presented to the Berlin Academy a functional whose infimum is 0 but is never attained [12, 13, 14]. It was a counterexample to the Dirichlet principle, the assumption that a minimising function always exists (note 1).

In 1900 Hilbert posed his 19th problem: are the solutions of regular variational problems always analytic [15, 16]? It was a question, not a result.

In the same years Hilbert repaired the Dirichlet principle [12, 13, 17, 18]. He took minimising sequences and used compactness to extract a limit, the start of what is now called the direct method. This was done in two papers, of 1900 and 1904, followed by a 1905 version in Crelle's *Journal für die reine und angewandte Mathematik* 129, 63–67 (note 1). The work rests on Weierstrass's 1870 objection and closes the gap he had exposed.

In 1902 Hurwitz proved the isoperimetric inequality for rectifiable curves using Fourier series [10, 11, 19]. The Wirtinger–Parseval inequality on the Fourier coefficients gives the bound directly, with no need to assume that an optimum exists. The paper appeared in the *Annales de l'École Normale Supérieure* 19. This Fourier proof does not use Steiner's; arXiv 2603.14663 treats it as separate. Steiner's existence gap was in fact filled by Weierstrass, around 1870 according to arXiv 2603.14663 and in 1879 according to Alama (note 2).

In 1903 or 1904 Sergei Bernstein gave the first partial answer to Hilbert's 19th problem [15, 16, 17]. He showed that C^3 solutions of analytic elliptic equations in two

variables are analytic, by means of a priori estimates. EoM dates this 1903; Wikipedia and Giaquinta give 1904, the year of his thesis (note 2).

Minimal surfaces and optimal control

In 1931 Jesse Douglas solved the Plateau problem: to find a minimal surface spanning a given closed curve [20, 21, 22]. He minimised what is now called the Douglas functional over parametrisations of the boundary. He received one of the first Fields Medals in 1936. Tibor Radó solved the problem independently, by minimising energy, for rectifiable curves (note 1). Both rest on the problem Plateau had posed.

In 1956 Pontryagin and his students formulated the maximum principle; the book followed in 1961 [23, 24, 25]. It states that an optimal control maximises the control Hamiltonian at each moment along the optimal trajectory. A quick check: for the double integrator $x'' = u$ with $|u| \le 1$, starting at rest at $x = 1$, the minimum-time control is bang-bang, switching once, and the time is $T = 2.0$. Wikipedia says the principle was derived using ideas from the classical calculus of variations, and EoM that the classical conditions, Weierstrass's included, follow from it [23, 25]. Neither names an earlier result that the proof uses, so the principle is treated here as a new start. Bellman's dynamic programming, from the same years, is a related approach (note 3).

Regularity, its limits, and soap bubbles

In 1956–57 Ennio De Giorgi, and in 1957–58 John Nash, proved independently that weak solutions of elliptic equations with merely bounded measurable coefficients are Hölder continuous [15, 26]. Applied to the derivatives of a minimiser with square-integrable derivatives, this makes its first derivatives Hölder continuous; with the earlier work of Bernstein and Schauder, this completes the answer to Hilbert's 19th problem. MacTutor quotes De Giorgi: "Nash and I proved the same theorem" [26]. The theorem proves Hilbert's 1900 conjecture, and MacTutor also credits De Giorgi with solving the problem. The interval is $1957 - 1900 = 57$ years.

In 1968 three sets of counterexamples marked the limit of this theorem [15, 27, 28]. De Giorgi, and independently Giusti and Miranda, gave discontinuous minimisers of elliptic systems; both papers appeared in the *Bollettino dell'Unione Matematica Italiana*. Maz'ya's examples, in *Funktsional'nyi Analiz* 2(3), concern single equations of order higher than two, not systems (note 4). All three rest on the De Giorgi–Nash theorem, showing where it stops.

In 2000 Hutchings, Morgan, Ritoré and Ros proved the double bubble conjecture in $\mathbb{R}^3$: the standard double soap bubble is the least-area way to enclose two given volumes [29, 30, 31]. The proof uses symmetry, a bound on the number of components and an instability argument; it was published in the *Annals of Mathematics* 155 (2002). It rests on Hutchings's 1997 structure theorem and bound on components, in the *Journal of Geometric Analysis* 7.

In 2025 Emanuel Milman and Joe Neeman published, in *Acta Mathematica* 234, proofs of the double, triple and quadruple bubble conjectures in $\mathbb{R}^n$ and S^n (arXiv 2022) [29, 32]. They use reflection symmetry, spherical interfaces and Voronoi-type structure. According to the AMR article on the result, the work uses Hutchings's 1997 structure theorem, and it proves the multi-bubble conjecture in these cases [29].

It is independent of the 2000 proof: AMR says the higher-dimensional cases “require completely new ideas”. Morgan notes that it depends on regularity theory that is still incomplete (note 7).

Notes

1. Euler’s *Methodus* is dated 1744 by the Library of Congress record and by MacTutor; Wikipedia’s “1750s” refers to the analytic form developed with Lagrange. Weierstrass’s objection dates from 1870, when he presented it to the Berlin Academy (Bottazzini, Wikipedia); EoM’s “late 1860s” is looser. Hilbert’s rescue of the Dirichlet principle consists of two papers, of 1900 and 1904 (Giaquinta); the 1905 Crelle paper is a later version. Radó solved the Plateau problem independently of Douglas (Wikipedia, FU Berlin).
2. Dates still differ between sources. Steiner’s symmetrisation is 1838 in Wikipedia and arXiv 2603.14663 but 1842 in Alama. Bernstein’s result is 1903 in EoM and 1904 in Wikipedia and Giaquinta. Weierstrass’s existence proof for the isoperimetric problem is c. 1870 in arXiv 2603.14663 and 1879 in Alama.
3. No source found states that Bellman’s dynamic programming influenced the discovery of the maximum principle. The history by Pontryagin’s own school attributes it to Pontryagin and his students at the Steklov department of differential equations [33].
4. Maz’ya’s 1968 examples are single higher-order equations, not systems; earlier versions of this chain had this wrong.
5. Lagrange’s Turin memoir is dated 1760 on MacTutor’s brachistochrone page; the DSB says it was written by 1759 and published in 1762.
6. Alama says Zenodorus took the polygon case as obvious; MacTutor says he proved it. MacTutor counts Johann Bernoulli’s 1697 solution as valid, while Wikipedia calls his derivation flawed.
7. The regularity theory under Milman and Neeman’s proof is incomplete, according to Morgan. Not covered in this chain: the second variation (Legendre, Jacobi), Weierstrass’s sufficient conditions, Tonelli’s direct method, Γ-convergence and Bellman.

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Convex and discrete geometry

Convex and discrete geometry studies convex bodies, packings of spheres and finite configurations of points. Its chain starts with Euclid's proof that there are exactly five regular solids (*Elements* XIII.18) and then runs in three lines. Packing goes from Kepler's 1611 conjecture through Gauss's lattice case and Fejes Tóth's finite reduction to Hales's proof (1998, *Annals* 2005) and its formal check. Alongside it, the Cohn–Elkies bound leads to Viazovska's proofs in dimensions 8 and 24, formally verified in Lean in February 2026.

Convexity runs from Brunn–Minkowski and Helly–Radon to Bourgain's slicing problem, solved by Klartag and Lehec (arXiv December 2024, *GAFA* 2025). A line of counterexamples runs from Borsuk's question (1933) to Kahn and Kalai (1993) and then to examples in 64 and, as claimed, 63 dimensions. Euclid's theorem and Thue's planar result stand on their own: no later proof in this chain is built on them. Euler's polyhedron formula belongs to the topology chain.

Regular solids and the packing problem

Around 300 BC Euclid proved that there are exactly five convex regular polyhedra [1, 2]. At each vertex the face angles must add up to less than four right angles, which leaves only five possibilities. Book XIII probably draws on Theaetetus (note 1). A quick check: the condition becomes $1/p + 1/q > 1/2$ for faces with p sides and q faces at a vertex, and this inequality has exactly 5 solutions in integers $p, q \geq 3$.

In 1611 Kepler conjectured that the face-centred cubic (fcc) packing, the way greengrocers stack oranges, is the densest packing of equal spheres [3, 4]. He gave no proof. The fcc density is $\pi/\sqrt{18} \approx 0.74048$.

In 1831 Gauss settled the case of lattice packings: among them, fcc is optimal [3, 4, 5, 6]. He did it by the reduction theory of ternary quadratic forms, in a review of a book by Seeber published in the *Göttingische gelehrte Anzeigen* on 9 July 1831. This rests on the problem Kepler had posed.

In 1890 Axel Thue announced that the hexagonal packing is the densest in the plane [4, 7, 8, 5]. His proof was printed in 1910, and the first fully rigorous proof came from László Fejes Tóth, in 1940 according to EoM and 1942 according to Wikipedia (note 2). The hexagonal density is $\pi/\sqrt{12} \approx 0.9069$. This planar result stands apart: it neither rests on earlier links here nor serves as the basis of a later one.

In 1900 Hilbert posed his 18th problem, which includes the sphere packing question [9, 4, 8]. It was a problem, not a result.

In 1953 Fejes Tóth reduced the Kepler conjecture to a finite optimisation over a cluster of spheres, with a correction term, which could in principle be minimised by computer [3, 4]. Hales's proof came $1998 - 1953 = 45$ years later.

In 1998 Thomas Hales, working with Samuel Ferguson, announced a proof of the Kepler conjecture; it appeared in the *Annals of Mathematics* 162 (2005), 1065–1185 [3, 10]. He followed Fejes Tóth's strategy, using a hybrid Voronoi–Delaunay

decomposition and linear programming by computer. The proof rests on Kepler's conjecture, on Fejes Tóth's 1953 reduction, and on Hilbert's 18th problem, which EoM also says includes sphere packing (note 3). It came 2005 − 1611 = 394 years after Kepler.

Because the proof was so heavily computer-based, Hales and many collaborators produced a formal proof in the HOL Light and Isabelle proof assistants (the Flyspeck project) [11, 3]. It was completed on 10 August 2014 and published in *Forum of Mathematics, Pi* in 2017. It rests on Hales's 2005 proof, and came 2014 − 1611 = 403 years after Kepler.

Convexity: Brunn–Minkowski, Helly and Radon

In 1887 Hermann Brunn, and in 1896 or 1897 Hermann Minkowski, proved what is now the Brunn–Minkowski inequality [12, 13, 14, 15]. The n-th root of volume is concave along Minkowski combinations of convex bodies, with equality exactly for homothetic bodies. Minkowski found an error in Brunn's proof, which Brunn corrected, and gave his own proof (note 4). The inequality quickly yields the isoperimetric inequality for convex bodies. A quick check with the boxes $[0,1]^2$ and $[0,4]\times[0,1]$: their sum is a 5×2 box, and $\sqrt{10} \approx 3.162 \geq \sqrt{1} + \sqrt{4} = 3$. This result does not rest on an earlier link in this chain.

In 1913 Eduard Helly found, and in 1923 he published, the theorem that bears his name [16, 17]. If every n+1 members of a finite family of convex sets in $\mathbb{R}^n$ have a common point, then all of them do. Dénes König found another proof independently in 1922.

In 1921 Johann Radon proved that any n+2 points in $\mathbb{R}^n$ can be split into two parts whose convex hulls meet [16, 17]. By induction this gives a new proof of Helly's theorem, so Radon's result rests on Helly's of 1913. A small example: (1, 1) lies in the triangle with vertices (0, 0), (4, 0) and (0, 4), so the four points split into that point and the triangle.

In 1986 Jean Bourgain posed the slicing problem [18, 19]. Does every convex body of volume 1, in any dimension, have a hyperplane section whose volume is above a fixed constant c?

In December 2024 Bo'az Klartag and Joseph Lehec gave a positive answer [18, 19, 20, 21]. They used M-ellipsoids, stochastic localisation with Guan's recent bound, and the stability of the Shannon–Stam inequality. The preprint appeared on arXiv on 19 December 2024; the paper was published in *Geometric and Functional Analysis* 35 (2025), 1147–1168. It proves Bourgain's conjecture of 1986, 2024 − 1986 = 38 years earlier, and rests on Guan's 2024 bound.

Sphere packing in high dimensions

In 2003 Henry Cohn and Noam Elkies published a linear-programming bound for sphere packing (arXiv 2001, *Annals* 157) [22, 23]. It is a sphere-packing analogue of the linear-programming bounds for error-correcting codes. They conjectured that it is sharp in dimensions 8 and 24. The bound does not rest on an earlier link in this chain.

In 2016 Maryna Viazovska proved that the E8 lattice gives the densest packing in R^8 [22, 23, 24, 25]. She constructed the "magic function" that Cohn and Elkies needed, from modular and quasimodular forms. The paper appeared in the *Annals* 185(3) (2017), 991–1015. It proves the Cohn–Elkies conjecture for n = 8. The E8 density is $\pi^4/384 \approx 0.25367$.

Within a week, Cohn, Kumar, Miller, Radchenko and Viazovska proved that the Leech lattice is densest in R^{24}, with an analogous magic function [26, 23, 25]. Their abstract describes the work as "building on Viazovska's recent solution". It appeared in the *Annals* 185(3) (2017), 1017–1033. The Leech lattice density is $\pi^{12}/12! \approx 0.0019296$.

In 2025 Klartag proved that there are lattice packings in dimension n of density at least $c \cdot n^2 \cdot 2^{-n}$ [27, 28, 29]. The proof lets an ellipsoid evolve randomly while it stays free of lattice points. The preprint appeared on arXiv on 7 April 2025, and the paper in *Inventiones Mathematicae* in 2026. It improves the classical bounds of Rogers (1947) and of Minkowski–Hlawka; Klartag's own text names these as the results improved, not as tools used in the proof. The gain over the earlier order n log n is n / ln n, about 21.7 at n = 100.

In February 2026 a Lean formalisation of the dimension 8 proof was verified [30, 31, 32]. The team, including Hariharan, Birkbeck and Viazovska, combined a human-written blueprint with autoformalisation by Math, Inc.'s system Gauss. The dimension 24 case followed within two weeks, in over 200,000 lines of code. Both formalisations rest on Viazovska's 2016 proof.

Counterexamples: Borsuk and Rupert

In 1933 Karol Borsuk asked whether every bounded set in R^n can be split into n+1 pieces of smaller diameter, and proved this for n = 2 [33, 34].

In 1993 Jeff Kahn and Gil Kalai showed that Borsuk's conjecture fails in high dimensions [34, 33]. The number of pieces needed, f(d), is at least 1.2^√d, which eventually exceeds d + 1. A quick check: the first d with 1.2^√d > d + 1 is 1652. This is a counterexample to Borsuk's 1933 question.

In 2014 Thomas Jenrich and Andries Brouwer gave a counterexample in dimension 64 [35, 33]. It is a two-distance set of 352 points inside Andriy Bondarenko's 65-dimensional example, on which it rests. It continues the line of counterexamples opened by Kahn and Kalai, though neither source says the construction uses their method.

In 2026 a counterexample in dimension 63 was claimed, with 321 points: a 320-point core plus one added point, generated with an AI model [36, 33]. There are two separate claims: Max Grinsztajn on GitHub in May 2026, according to Wikipedia, and Yibo Ji on arXiv on 12 August 2026. Ji's abstract states that the construction starts from the $G_2(4)$ Euclidean representation and the Jenrich–Brouwer 320-point core in dimension 63, and adds one projected and rescaled point [36]. Neither claim was refereed as of 29 October 2026, and priority is disputed (note 5).

A different question concerns passing a solid through a hole in a copy of itself. In 2017 Jerrard, Wetzel and Yuan conjectured, hesitantly, that every convex polyhedron has this Rupert property [37]. In August 2025 Jakob Steininger and Sergey Yurkevich described the Noperthedron, with 90 vertices and 152 faces, which has no Rupert

passage [38, 39]. This refutes the 2017 conjecture. It is still a preprint (v2 January 2026). A quick check with Euler's formula: it has $E = 90 + 152 - 2 = 240$ edges.

Notes

1. Britannica credits Euclid with the first known proof that there are five regular solids; Wikipedia says Theaetetus may have given it.
2. Fejes Tóth's rigorous proof of Thue's theorem is dated 1940 in EoM and 1942 in Wikipedia. Thue's 1890 announcement is attested by Hales and Wikipedia.
3. Wikipedia credits Hales alone with the 1998 announcement, while Hales's *Notices* article names Ferguson as co-author.
4. Minkowski's proof is dated 1896 by Wikipedia and 1897 by EoM; Gardner gives no year. That Brunn–Minkowski yields the isoperimetric inequality for convex bodies is stated by Gardner.
5. Wikipedia credits the 63-dimensional example to Max Grinsztajn (GitHub, May 2026), while arXiv 2608.12561 (12 August 2026) is by Yibo Ji; they appear to be two separate claims. Ji's abstract says the example and proof were generated entirely by an AI model and that he verified them himself. Both claims are unrefereed.

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Global analysis

Global analysis studies analysis on whole manifolds: how functions, differential forms and differential operators reflect the shape of the space they live on. The field has no ancient roots. It starts in the 1920s with Morse's theory of critical points, Cartan's suggestion that differential forms detect topology, de Rham's theorem (1931) and Hodge's harmonic integrals, whose 1936 proof had a gap that Weyl and Kodaira repaired.

Three lines follow. Index theory runs from Gelfand's 1960 problem through Atiyah–Singer to the heat-equation and Getzler proofs. Morse and Floer theory runs from Floer's cases of the Arnold conjecture to the proofs with integer coefficients of 2022. Gauge theory brings the Seiberg–Witten invariants and the Thom conjecture (1994). The newest step, in 2026, is the claimed proof of the full Arnold–Givental conjecture by Bai, Shelukhin, Wang and Xu, an unrefereed preprint of August 2026.

Many of the dependencies below are stated only by Wikipedia; they are given here as stated there. The Atiyah–Bott fixed-point theorem stands apart: no dated source names an earlier result in this chain on which its proof is built.

Foundations: Morse, de Rham and Hodge

In 1925 Marston Morse found relations between the critical points of a function on a manifold and the manifold's topology [1, 2]. Critical points are counted by their index, and the number c_k of index k is at least the k-th Betti number b_k: these are the Morse inequalities. His book *The calculus of variations in the large* followed in 1934. A quick check: the function cos u + cos v on the torus has one minimum, two saddles and one maximum, so c = (1, 2, 1). The alternating sum $1 - 2 + 1 = 0$ equals the Euler characteristic of the torus, and each c_k ≥ b_k = (1, 2, 1).

In 1928 Élie Cartan suggested that differential forms and Betti numbers are linked [3, 4]. It was a suggestion, without proof.

In 1930 W. V. D. Hodge applied Lefschetz's topological ideas to algebraic varieties [5, 3]. This rests on Lefschetz's work of 1929.

In 1931 Georges de Rham proved, in his thesis, that the cohomology defined by differential forms equals singular cohomology [3, 4, 6]. The isomorphism is given by integrating forms over chains. It proves Cartan's conjecture; Bott, quoted by MacTutor, says Cartan had conjectured it and even used it in 1928 [4].

Between 1933 and 1941 Hodge proved that every de Rham class has a unique harmonic representative [3, 5, 7]. The proof uses the elliptic theory of the Laplacian acting on forms. There were attempts in 1933 and 1936, and a full account in 1941 (note 1). A quick check of the counting: on a surface of genus 3 the space of harmonic 1-forms has dimension $b_1 = 2g = 6$. The work rests on de Rham's theorem; Struik's 1944 review notes that Hodge's 1941 book cites de Rham's 1931 paper [7].

In 1942–43 Hermann Weyl and Kunihiko Kodaira independently repaired a flaw in Hodge's proof that Frederic Bohnenblust had found [3, 8, 7]. Weyl's repair appeared

in the *Annals of Mathematics* 44 (1943). Both rest on Hodge's 1936 proof and close its gap (note 1).

Index theory

In 1960 Israel Gelfand asked for a topological formula for the index of an elliptic operator: the dimension of its kernel minus that of its cokernel [9, 10]. It was a problem only.

Michael Atiyah and Isadore Singer answered it: the analytic index equals a topological index [9, 11, 12, 13]. A proof sketch using cobordism was announced in 1963; the authors never published it, and it appeared only in Palais's book (note 3). A proof by K-theory appeared in the *Annals* from 1968. The Riemann–Roch theorem, Hirzebruch's signature theorem and the Gauss–Bonnet theorem are special cases. Two quick checks: for a line bundle of degree $d = 5$ on a curve of genus $g = 2$, Riemann–Roch gives index $d - g + 1 = 4$; and Gauss–Bonnet on the sphere gives $(1/2\pi)\int K\,dA = 2 = \chi(S^2)$. The theorem answers Gelfand's 1960 problem, and the 1968 proof rests on the K-theory Atiyah had developed with Hirzebruch.

Between 1966 and 1968 Atiyah and Raoul Bott proved a Lefschetz fixed-point formula for elliptic complexes [14, 15, 11]. A simple instance of a fixed-point index: the map $z \mapsto z^2$ has index 2 at 0, its winding number. MacTutor says the index theorem combined with symmetry led to this result; Wikipedia says its history is entangled with the index theorem and began with a conjecture of Shimura at Woods Hole in 1964. No source names an earlier result that the proof itself uses, so it is treated here as a separate development.

In 1973 Atiyah, Bott and Vijay Patodi gave a new proof of the index theorem [9, 16]. It uses the small-time asymptotics of the heat kernel, where terms cancel locally and leave only the topological index. It is a new proof of Atiyah–Singer.

In 1983 and 1986 Ezra Getzler gave further new proofs, through the local index theorem for Dirac operators [9, 12, 17]. The first used pseudodifferential operators on supermanifolds, in *Communications in Mathematical Physics* 92 (1983); the second was "a short proof", in *Topology* 25 (1986). Freed describes Getzler's work as a new proof of the local index theorem of Patodi and Gilkey, so it builds on the heat-equation approach of Atiyah, Bott and Patodi; Getzler's own titles name only Atiyah–Singer [12, 17].

Morse theory, Floer theory and the Arnold conjecture

In 1982 Edward Witten gave an analytic proof of the Morse inequalities, through the de Rham complex of a perturbed operator [1, 18, 19]. Atiyah, quoted by MacTutor, called it "a brilliant proof of the classic Morse inequalities". It appeared in the *Journal of Differential Geometry* 17 (1982) (note 2). It is a new proof of Morse's 1925 result.

Between 1987 and 1989 Andreas Floer created Floer homology [20, 21, 22, 23]. It is an infinite-dimensional Morse homology of the action functional on the loop space, whose flow lines are pseudoholomorphic curves. With it he proved cases of the Arnold conjecture, which bounds the number of fixed points of a Hamiltonian map from below by the sum of the Betti numbers. A quick check: for the torus T^(2n) that sum is

2^(2n), giving 4, 16 and 64 for n = 1, 2, 3. Floer announced the work in the *Bulletin of the AMS* 16 (1987) and gave the general proof in 1988–89 (note 4).

Floer's theory rests on Morse's: his 1987 note is titled "Morse theory for fixed points", and Wikipedia calls it an infinite-dimensional analogue of Morse homology [20, 22]. It also rests on Gromov's 1985 compactness theorem, which makes the counts of flow lines finite; Floer's 1987 note cites Gromov's paper (note 3). And it proves a special case of the Arnold conjecture (note 5).

In 1989 Alexander Givental proved a special case of the Lagrangian version, the Arnold–Givental conjecture [24, 25, 26]. Lu's paper gives Givental's case as the pair (CP^n, RP^n), without a year. This result does not rest on an earlier link here.

In 1999 Kenji Fukaya and Kaoru Ono proved the weak Arnold conjecture with rational coefficients for all closed symplectic manifolds, using Kuranishi structures, in *Topology* 38, 933–1048 [24, 27, 28]. Bai and Xu's history describes the result as the bound by rational Betti numbers. It proves the homological version of the conjecture over the rationals.

In 2021 Mohammed Abouzaid and Andrew Blumberg proved the Arnold conjecture in Morava K-theory, using Floer homotopy theory (arXiv, 2 March 2021) [24, 29]. It is a further case of the conjecture.

In 2022 Shaoyun Bai and Guangbo Xu, and independently Semon Rezchikov, proved the Arnold conjecture over the integers, torsion included [24, 27, 30]. Bai and Xu used Hamiltonian Floer theory over an integer Novikov ring with global Kuranishi charts (18 September 2022); Rezchikov's proof followed four days later (22 September 2022). Both rest on the Arnold conjecture, now proved over Z, and on the global charts of Abouzaid, McLean and Smith. Neither had a journal reference as of 29 October 2026.

In August 2026 Bai, Egor Shelukhin, Wang and Xu claimed the Arnold–Givental conjecture in full generality [26, 24]. They combine integral Floer theory, Lu's reduction and a new Z/2-equivariant localisation. The preprint (arXiv, 27 August 2026, v1 only) was unrefereed as of 29 October 2026. It extends Givental's special case to a claimed full proof. According to its own abstract it uses Bai and Xu's integral Floer theory of 2022; no other source says so.

Gauge theory and the Thom conjecture

In 1994 Nathan Seiberg and Edward Witten introduced invariants of 4-manifolds from counting solutions of the abelian monopole equations, in place of the SU(2) instantons used by Donaldson [31, 32]. Witten's arXiv paper is dated 15 November 1994. It reformulates Donaldson's instanton invariants; the abstract calls the new invariants equivalent.

In 1994 Peter Kronheimer and Tomasz Mrowka proved the Thom conjecture: smooth algebraic curves in CP^2 minimise genus in their homology class [33, 34]. The monopole invariants, they wrote, "lead quickly to a proof". The proof rests on the Seiberg–Witten invariants (note 6).

In 2000 Peter Ozsváth and Zoltán Szabó proved that symplectic surfaces in symplectic 4-manifolds are genus minimising, which implies the Thom conjecture [34, 35]. According to the *Annals* abstract and the paper itself, the proof uses the Seiberg–Witten invariants together with Taubes's theorems. It generalises Kronheimer and

Mrowka's result, but no source says it uses their proof.

Notes

1. MacTutor calls Hodge's 1941 *The theory and applications of harmonic integrals* a book, while Wikipedia cites a 1941 *Nature* article of that title; Struik's 1944 review confirms the book. Wikipedia places the flaw in the 1936 proof, the DSB in the 1941 book.
2. MacTutor dates Witten's Morse paper 1984; the journal record gives *J. Diff. Geom.* 17 (1982), 661–692.
3. The 1963 Atiyah–Singer sketch was never published by the authors (Wikipedia, Freed). Floer's use of Gromov's 1985 compactness is stated by Wikipedia and shown by the 1987 note's own reference list.
4. Floer's work is a 1987 announcement (*Bull. AMS* 16, 279–281, MacTutor) and a general proof in 1988–89 (Wikipedia gives 1988, Zehnder 1989).
5. Arnold posed his conjecture in 1965 (Bai and Xu cite Arnold 1965) or 1966 (Zehnder: first in 1966, at the ICM).
6. Kronheimer and Mrowka's proof is dated October 1994, a month before Witten's arXiv paper of 15 November 1994, so the theory circulated before it was posted.
7. Not covered in this chain: Conley–Zehnder, Lusternik–Schnirelmann, Palais–Smale and Eells–Sampson. Bai–Xu and Rezchikov have no journal reference, and the 2026 claim is unrefereed.

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Computer science

Theoretical computer science asks what can be computed, how fast, with how much memory, and how securely. Its chain starts with Euclid's algorithm for the greatest common divisor (c. 300 BC) and Lamé's 1844 bound on its number of steps. From Turing's 1936 machine it runs along three lines: automata (McCulloch–Pitts, Kleene, Rabin–Scott), complexity (Hartmanis–Stearns, Cook–Levin, the PCP theorem, Dinur) and public-key cryptography (Diffie–Hellman, then RSA).

The AKS primality test (2002), Babai's graph isomorphism result (2015/2017) and the learning theory of Valiant and of Blumer, Ehrenfeucht, Haussler and Warmuth (1984–1989) add side lines. In 2025 Williams fitted a computation of time t into about √t space, and Duan and colleagues beat Dijkstra's bound for directed shortest paths; both papers won best-paper awards at STOC. Lamé's analysis and the complexity theory of Hartmanis and Stearns are independent: no source shows that the later work used his result.

Algorithms before computers

Around 300 BC Euclid described the algorithm for the greatest common divisor (*Elements* VII.1–2) [1, 2]. Replace the larger number by its remainder on division by the smaller, and repeat. The remainders strictly decrease, so the process ends, and the last nonzero remainder divides both numbers. A quick check: gcd(144, 89), for two consecutive Fibonacci numbers, takes 10 division steps. The algorithm may be older than Euclid (note 1).

In 1844 Gabriel Lamé proved that Euclid's algorithm needs at most five times as many steps as the smaller number has decimal digits [1, 3, 4, 5]. Consecutive Fibonacci numbers are the worst case, since n steps force the smaller number b to be at least F(n+1). A quick check: over all pairs $1 \le b \le a \le 2000$, the number of steps never exceeds 5 times the digits of b; and the smallest pairs needing 2 to 8 steps are (3, 2), (5, 3), (8, 5), (13, 8), (21, 13), (34, 21) and (55, 34). This rests on Euclid's algorithm (note 2).

Machines and automata

In 1936 Alan Turing defined the Turing machine and proved that the Entscheidungsproblem is undecidable (*Proc. London Math. Soc.* (2) 42, 230–265) [6, 7]. The full chain for this result is told in the chapter on logic and computation.

In 1943 Warren McCulloch and Walter Pitts described nerve nets: networks of all-or-none neurons, each treated as a unit of propositional logic (*Bulletin of Mathematical Biophysics* 5, 115–133, December 1943) [8, 9]. This is the model from which Kleene later started. It does not rest on an earlier link here.

In December 1951 Stephen Kleene showed, in RAND memorandum RM-704, that the events a finite automaton or McCulloch–Pitts nerve net can respond to are exactly the regular events [8, 10]. He defined these by what are now called regular expressions.

The result rests on McCulloch and Pitts's nets; a book version appeared in *Automata Studies* (1956) (note 3).

In 1959 Michael Rabin and Dana Scott proved that nondeterministic finite automata recognise exactly the same languages as deterministic ones [11, 12, 13]. The subset construction takes as states of the deterministic automaton the sets of states of the nondeterministic one. A quick check: for the language "the n-th symbol from the end is 1", an NFA with n + 1 states yields exactly 2^n reachable DFA states, for n = 1 to 8. The paper was written in 1957 and published in 1959. Rabin and Scott re-prove Kleene's theorem with nondeterministic automata; this is an influence, not a step the proof depends on.

Shortest paths, complexity and the PCP theorem

In 1956 Edsger Dijkstra conceived his shortest-path algorithm, and he published it in 1959 (*Numerische Mathematik* 1, 269–271) [14, 15, 16]. It repeatedly settles the nearest unsettled vertex, which requires keeping the vertices in sorted order. With the Fibonacci heaps of Fredman and Tarjan (1984) it runs in time $\Theta(m + n \log n)$ on a graph with n vertices and m edges. It does not rest on an earlier link here.

In 1965 Juris Hartmanis and Richard Stearns defined time complexity classes and proved the time hierarchy theorem: with more time, strictly more problems can be solved [17, 18, 19, 20]. The proof diagonalises over multitape Turing machines that carry a time-bounded clock. It rests on Turing's machine and on the diagonal method, as the paper itself (Theorem 9) and Fortnow state.

In 1971 Stephen Cook introduced NP-completeness at STOC; Leonid Levin found it independently in 1973 [21, 22]. The full chain is in the chapter on logic and computation (note 6).

In 1992 Sanjeev Arora and Shmuel Safra, and Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan and Mario Szegedy, proved the PCP theorem, NP = PCP(log n, O(1)) [23, 24]. A proof of membership in NP can be written so that a verifier reads only a few randomly chosen bits of it. As a result, even finding approximate solutions to many problems stays NP-hard. The journal version appeared in *JACM* 45 (1998).

The PCP theorem is a statement about NP and so builds on the Cook–Levin notion, though no source states that the proof uses that theorem. It is part of a series of results honoured together by the 2001 Gödel Prize, including Feige, Goldwasser, Lovász, Safra and Szegedy (1991), according to SIGACT and Wikipedia [23, 24]. According to Wikipedia, it also extends the methods of Babai, Fortnow and Lund (1990) [24].

In April 2005 Irit Dinur gave a new, self-contained combinatorial proof of the PCP theorem (ECCC TR05-046, 17 April 2005; *JACM* 2007) [25, 24]. Her key lemma doubles the satisfiability gap of a constraint system whose underlying graph is an expander; iterating it yields the theorem. It reproves the PCP theorem without depending on the algebraic proof.

Cryptography, learning and primality

In 1976 Whitfield Diffie and Martin Hellman introduced the idea of public-key cryptography and a key exchange [26, 27]. Each party raises a common base g to a secret exponent; both can then compute the shared key g^(ab) mod q, while an

eavesdropper would have to solve a discrete logarithm. A quick check: with p = 23, g = 5, a = 6 and b = 15, both sides obtain the key 2. The idea does not rest on an earlier link here (note 5).

In 1977 Ronald Rivest, Adi Shamir and Leonard Adleman devised RSA encryption and signatures [28, 29, 30, 31]. If ed ≡ 1 (mod φ(n)), then Euler's theorem gives $(M^{e)}d \equiv M$ (mod n). A quick check: with n = 3233, e = 17 and d = 2753, the message 65 encrypts to 2790 and decrypts back to 65; and 65^3120 ≡ 1 (mod 3233), where 3120 = φ(3233). The paper was received on 4 April 1977 and appeared in *Communications of the ACM* 21(2), February 1978 (note 4). It rests on the Diffie–Hellman concept and on Euler's theorem from number theory: the paper uses "an identity due to Euler and Fermat", and Wikipedia gives the same proof, with a variant using Fermat's little theorem and the Chinese remainder theorem.

In 1984 Leslie Valiant defined the PAC ("probably approximately correct") model of learning (*CACM* 27(11)) [32, 33, 34]. An algorithm learns a class if, in time polynomial in an adjustable parameter h and the number of variables, it outputs with probability at least 1 − 1/h a hypothesis whose error is at most 1/h. The two roles of 1/h were later split into separate parameters δ and ε (note 3). The model does not rest on an earlier link here.

In 1989 Anselm Blumer, Andrzej Ehrenfeucht, David Haussler and Manfred Warmuth proved that a concept class is uniformly learnable if and only if its Vapnik–Chervonenkis dimension is finite, and that any consistent hypothesis then works [34, 35, 33, 36]. A quick check of their sample bound, with logarithms to base 2: VC dimension d = 10, ε = 0.1 and δ = 0.05 give m ≈ 5618 examples. The result rests on Valiant's model, as the paper's abstract and Wikipedia state, and on Vapnik and Chervonenkis's uniform convergence from probability and statistics, as the paper and the Brown CS155 notes state.

In 2002 Manindra Agrawal, Neeraj Kayal and Nitin Saxena proved that PRIMES is in P [37, 38, 39]. The number n is prime if and only if $(X + a)^n \equiv X^n + a$ (mod n); they showed it suffices to check this modulo X^r − 1 for one small r and a few values of a. A quick check: with a = 1 the full identity holds exactly for the primes from 2 to 399, and the Carmichael number 561 passes Fermat's test to base 2 but fails the identity. The test rests on Fermat's little theorem, which the identity generalises, as the AKS paper and Wikipedia say. The AKS paper also cites the randomised test of Agrawal and Biswas (*JACM* 50, 2003), based on the same identity; only the AKS paper states this link.

Graph isomorphism and the results of 2024–25

In 2015 László Babai announced that graph isomorphism can be decided in quasipolynomial time [40, 41, 42]. He built on Luks's framework of 1983, adding group-theoretic local certificates and combinatorial canonical partitioning. Harald Helfgott then found an error in the analysis of the "Split-or-Johnson" step. Babai found a replacement on 7 January 2017 and posted it on 9 January. The work improves on Luks's result and draws on group theory, as the arXiv abstract and the CACM survey state (note 6).

In 2024 James Cook and Ian Mertz solved Tree Evaluation in space O(log n · log log n) [43, 44, 45]. This improved their own 2020 bound of $O(\log^2 n / \log \log n)$. The

preprint is ECCC TR23-174 (15 November 2023), and the paper appeared at STOC 2024, according to Williams' arXiv abstract and the MIT record.

In 2025 Ryan Williams proved that TIME[t] ⊆ SPACE[O(√(t log t))] [46, 47, 45]. He reduced a time-t computation to instances of Tree Evaluation and solved those with the Cook–Mertz algorithm. A quick check of the scale: for $t = 10^{12}$, the old bound t/log t is about 2.5×10^{10} memory cells, the new bound √(t log t) about 6.3×10^{6}. The result rests on Cook–Mertz, according to the paper and Fortnow's CACM post. It improves the bound of Hopcroft, Paul and Valiant, of t/log t space (note 6).

In 2025 Ran Duan, Jiayi Mao, Xiao Mao, Xinkai Shu and Longhui Yin found directed shortest paths in time O(m log^(2/3) n) [48, 14, 49]. This is the first bound to beat O(m + n log n) on sparse graphs. Quanta and Tsinghua say that the method "combines the ideas of Bellman-Ford algorithm and the recursive partial ordering method", so it rests on Bellman–Ford relaxation from optimisation. It improves on Dijkstra's bound without using Dijkstra's method. A quick check: at $n = 10^{9}$ and m = 4n, the old bound is still the smaller one (ratio 0.88), so the gain is only asymptotic.

Notes

1. Euclid's algorithm may already have been known to Eudoxus, and Brezinski credits Theaetetus; only Wikipedia states this. Britannica names only Euclid.
2. Lamé's date is 1844: his note is in *Comptes rendus* 19, 867–870 (Fibonacci Quarterly, IREM Grenoble). The Prime Pages' 1845 ("In 1845 G. Lame' proved") matches a later note by Nievengloski in *Nouvelles annales* 4 [50, 51].
3. Valiant's paper is *CACM* 27(11), November 1984; the ACM Turing Award page says 1983 [52]. Kleene's memo is from December 1951; the book version is *Automata Studies* (1956), pp. 3–42 [53].
4. RSA was received on 4 April 1977 and revised on 1 September 1977, according to the paper itself [54], and appeared in *CACM* 21(2), February 1978. The MIT bibliography entry and Wikipedia give no date of receipt.
5. On priority in public-key cryptography: Clifford Cocks found the RSA scheme at GCHQ in 1973, and it was made public in 1997 [55, 56]. A claim of earlier discovery at the NSA rests on a single source.
6. Dijkstra conceived his algorithm in 1956 and published it in 1959; the PCP theorem has conference dates of 1992 and a journal date of 1998. Babai's paper has no refereed journal version (arXiv v2 has no journal reference), and whether graph isomorphism is in P is open [57, 42]. Hopcroft–Paul–Valiant is dated 1975 in Williams' abstract and 1977 in Fortnow's post: 1975 is the FOCS conference, 1977 the journal. ACM's Cook page dates the NP-completeness paper to STOC 1971 but also calls it "the 1970 paper"; Wikipedia gives 1971. Shor's 1994 algorithm lies outside this chain, and P versus NP is open (see the logic chapter).

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Mechanics

Mechanics, in the mathematical sense, studies the motion and equilibrium of particles and systems of bodies under forces. The chain starts with Archimedes' statics (c. 250 BC). It runs through d'Alembert (1743) and Lagrange (1788) to Hamilton (1834) and Jacobi (1837). From there one line goes through Noether (1918) to symplectic reduction (1974) and Gromov's non-squeezing theorem (1985).

Two lines on the n-body problem reach the present. Painlevé's question (1895) was closed by Xia for $n \geq 5$ (1992) and by Xue for $n = 4$ (*Acta Mathematica* 2020). Chazy's final motions (1922) lead through Sitnikov and Alekseev to Guardia, Martín, Paradela and Seara (arXiv 2022, still a preprint). Liouville's theorem of 1838 does not rest on an earlier link here, and Xia and Gerver worked independently of each other. The line from Kepler and Newton through Poincaré to KAM theory is told in the chapter on differential equations and dynamics and is not repeated here.

From the lever to analytical mechanics

Around 250 BC Archimedes proved the law of the lever and found the centres of gravity of plane figures (*On the Equilibrium of Planes* I.6–7) [1, 2, 3]. He started from postulates about equal weights at equal distances. What exactly the proof establishes is debated among historians. A quick check: weights 3 and 5 at distances 5 and 3 give equal torques, $3 \cdot 5 = 5 \cdot 3 = 15$, so the lever balances. This result does not rest on an earlier link; the primary text is available in MacTutor's English translation of Book I [2].

In 1743 Jean d'Alembert published his *Traité de dynamique* [4, 5, 6]. His principle adds $-m \cdot a$ to the forces as an extra "force of inertia", which turns a problem of dynamics into one of statics. The forces of constraint do no virtual work and drop out of the equations. It does not rest on an earlier link here.

In 1788 Joseph-Louis Lagrange published the *Mécanique analytique* [7, 6, 8]. It gives the general equation of dynamics and Lagrange's equations in generalised coordinates, and, as he boasted, it contains "no figures". A quick check: for a pendulum, $L = \tfrac{1}{2}ml^2\theta^2 + mgl\cos\theta$ gives $\theta'' + (g/l)\sin\theta = 0$. The derivation uses d'Alembert's principle of 1743. According to MacTutor, Archimedean statics shaped the static case, which Lagrange reduced to a single general formula through the principle of virtual displacements (note 1) [8, 9].

Hamilton, Jacobi and Liouville

In 1834–35 William Rowan Hamilton published his "General Method in Dynamics" in the *Philosophical Transactions* [10, 11, 12]. The characteristic function from his ray optics of 1826–32 becomes the action taken between initial and final configurations. The method rests on his own optics, as MacTutor and the 1834 paper confirm. The paper also cites the *Mécanique analytique* and says that Lagrange "has perhaps done more than any other analyst"; it builds on Lagrange's work, though only the paper itself says so [12].

In 1837 Carl Jacobi showed that a complete integral S(t, x, α) of the Hamilton–Jacobi equation solves the canonical equations, through ∂S/∂x = p and ∂S/∂α = β [13, 14, 11]. The function S generates a canonical transformation. The proof uses Hamilton's 1834 work (note 2).

In 1838 Joseph Liouville published a "Note sur la théorie de la variation des constantes arbitraires" (*Journal de Mathématiques Pures et Appliquées* 3, 342–349) [15, 16, 17]. He showed that a certain determinant u attached to the solutions satisfies du/dt = u·Σ ∂Pᵢ/∂xᵢ. In his words, whenever this sum is zero, "le dénominateur u est alors indépendant de t": for a divergence-free flow the Jacobian is constant, and equal to 1 when the constants are the initial values [15]. Henriksson credits this note with the mathematical background of Liouville's theorem, and Gibbs (1902) with its physical reading as conservation of "extension-in-phase" (note 3). A quick check: for the pendulum flow over a time T = 7.3, the determinant of the derivative of the flow map is 1.00000000. This result stands apart from the other links in the chain.

The n-body problem: singularities without collision

In 1895, in lectures published in 1897, Paul Painlevé proved that for three bodies every singularity of the motion is a collision [18, 19]. He showed that the minimum mutual distance tends to zero at a singularity, according to Diacu. He conjectured that for n ≥ 4 there are singularities without collisions. This does not rest on an earlier link here (note 4).

In 1908 Hugo von Zeipel showed that at a singularity without collision some body must reach infinity in finite time [18, 19]. The key fact is that at a collision singularity the moment of inertia has a limit. The work rests on the problem Painlevé had posed.

In 1975 John Mather and Richard McGehee found a singularity without collision in the collinear four-body problem [18, 19]. It occurs only after infinitely many binary collisions, which have to be regularised, so it was not accepted as a proof of Painlevé's conjecture. It rests on Painlevé's problem.

In 1984 Joseph Gerver proposed a mechanism [20, 18]. A light body on a hyperbolic Kepler orbit takes energy from an elliptic pair, and the encounter repeats in a self-similar way. He gave a heuristic planar five-body scenario (1984), a proof for the planar 3N-body problem (1991, *Journal of Differential Equations* 89) and a heuristic planar four-body model (2003, *Experimental Mathematics*). It is this last model that Xue later realised. Gerver's work is independent of Xia's.

In 1988, in his thesis, and in the *Annals* in 1992, Zhihong Xia proved that singularities without collision exist in the spatial five-body problem [18, 19]. Two eccentric binaries move in parallel planes, rotating in opposite senses, while a small fifth body oscillates along the axis between them. According to Diacu, the motion becomes unbounded in finite time. This proves Painlevé's conjecture for n ≥ 5 (note 5).

In 2014, with publication in *Acta Mathematica* 224(2) (2020), 253–388, Jinxin Xue proved that singularities without collision exist in a planar four-body problem, the last open case of Painlevé's conjecture [20, 18, 21]. He iterated Gerver's energy-extraction model, synchronised the two light bodies, and found a Cantor set of initial data that leads to the singularity. The proof uses Gerver's 2003 model: Xue works "following [G2]", and Wikipedia says it is "based on a model of Gerver". According to Xue's own paper, it also uses von Zeipel's 1908 theorem. It proves Painlevé's conjecture for n =

4, 2014 − 1895 = 119 years after it was posed, or 125 years counting to the journal publication of 2020.

The n-body problem: final motions and choreographies

In 1922 Jean Chazy classified the possible final motions, future and past, of the three-body problem [22, 23]. One of his classes is the oscillatory class OS, where the bodies come back again and again without settling into any limit. It does not rest on an earlier link here.

In 1960 Kirill Sitnikov gave the first rigorous oscillatory motions, in *Doklady Akademii Nauk SSSR* 133, 303–306 [22, 23, 24, 25]. In his restricted problem a massless body oscillates on the axis of a binary. He showed that every combination of past and future final motions occurs. It rests on the question Chazy had posed (note 6).

In the 1960s Vladimir Alekseev extended Sitnikov's example to the full, non-restricted three-body problem [26, 23]. Jürgen Moser later re-proved Sitnikov's results using Smale horseshoes, that is, symbolic dynamics. Alekseev's proof uses and extends Sitnikov's.

In 2000 Alain Chenciner and Richard Montgomery proved the existence of the figure-eight choreography, in which three equal masses chase each other along one figure-eight curve (*Annals* 152, 881–901) [27, 28, 29]. They minimised the action over a symmetry class of loops and showed that the minimiser has no collisions. A quick check with Simó's initial data and period T = 6.32591398: after one period the state returns to within about 7.5×10^{-8}, the angular momentum is 0, and after T/3 the positions are a permutation of the starting ones. The proof does not rest on an earlier link in this chain. Cristopher Moore had found the orbit numerically in 1993 (*Physical Review Letters* 70); the paper cites him in a note added in proof, an influence rather than a step of the proof.

In 2022 Marcel Guardia, Pau Martín, Jaime Paradela and Tere Seara proved the existence of oscillatory motions, and of hyperbolic invariant sets with symbolic dynamics, in the three-body problem for all masses except three equal ones [30, 22, 31]. As of 29 October 2026 it is still a preprint: arXiv 2207.14351 v1 has no journal reference, and Guardia's own page lists it under preprints. It builds on the method of Alekseev and Moser, a link inferred from the content: the sources that describe it come from the same research group. It comes 2022 − 1922 = 100 years after Chazy.

Symmetry and symplectic geometry

In 1918 Emmy Noether proved that every continuous symmetry of an action gives a conservation law [32, 33]. The proof combines the calculus of variations with Lie's groups. A quick check: for a Kepler orbit, which is invariant under rotation, the angular momentum L_z stays between 1.2000000000 and 1.2000000000 along the computed orbit. The proof uses Lagrange's 1788 formalism, and Lie groups from the chapter on topological groups.

In 1974 Jerrold Marsden and Alan Weinstein introduced symplectic reduction [34, 35, 36, 37]. Take the level set $\mu^{-1}(0)$ of the conserved momentum map and divide by the symmetry group: the result is again a symplectic manifold. The construction uses Noether's 1918 theorem, since the momentum map collects the conserved quantities.

Kenneth Meyer found it independently in 1973. Marsden and Weinstein call his paper "closely related work", and Wikipedia says the reduction is "due to K. Meyer (1973) and independently J.E. Marsden and A. Weinstein (1974)".

In 1985 Mikhail Gromov proved the non-squeezing theorem in his paper "Pseudo holomorphic curves in symplectic manifolds" [38, 39]. A ball $B^{2n}(r)$ embeds symplectically into the cylinder $B^2(R) \times \mathbb{R}^{2n-2}$ only if $r \leq R$. A quick check in the linear case: over 2000 random symplectic maps of $\mathbb{R}^4$, the smallest squared radius of a disc containing the projected unit ball was 1.09 in one run, never below 1; the volume-preserving but non-symplectic map diag(0.2, 5, 0.2, 5) squeezes it to 0.04 (note 7). The theorem sharpens Liouville's of 1838: symplectic maps preserve volume, but they are far more rigid than volume-preserving maps.

Notes

1. That Archimedean statics influenced Lagrange's treatment rests on MacTutor alone. The Encyclopedia of Mathematics traces the principle of virtual displacements through Ubaldi, Galileo, Wallis, Descartes and Johann Bernoulli to Lagrange, without naming Archimedes [9].
2. Jacobi's dates: EoM gives 1837 for his paper; Britannica dates the *Vorlesungen über Dynamik* to 1866 (lectures of 1842–43), and MacTutor names lectures of 1847–48. These are different events, but which lecture series is meant remains unclear.
3. The phase-space reading of Liouville's theorem is Gibbs's: Nash cites Gibbs 1884, Henriksson Gibbs 1902.
4. Diacu says Painlevé's proof for $n = 3$ used the minimum mutual distance, not the moment of inertia, which enters only with von Zeipel.
5. Xia's date: Diacu gives 1988 (the thesis), other sources 1992 (the *Annals*); both are right for different events. Diacu (1993) still calls the *Annals* paper "in press".
6. Sitnikov's date is 1960, *Dokl. Akad. Nauk SSSR* 133, 303–306 (Wikipedia, MAI); the IAS text's "in the 50's" is looser.
7. The radius found depends on how the random maps are drawn; the theorem guarantees only that it is never below 1.
8. The Hamilton–Jacobi analogy between classical and wave mechanics has a second source, Marmo, Morandi and Mukunda [40], but neither source calls it Schrödinger's own route. The scope of MSC 70 is "Mechanics of particles and systems" [41, 42]. No result from 2025–26 in this field was found in two sources.

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Elasticity and solids

The mathematical theory of elasticity describes how solid bodies deform under load and spring back. Its chain runs from Hooke's law (1660, printed 1678) through the elastica of the Bernoullis and Euler to the field equations of Navier (1821) and Cauchy (1822), with Germain's plate theory repaired by Kirchhoff (1850).

Its 20th-century lines are Korn's inequality for linear elasticity, Morrey's quasiconvexity and Ball's polyconvexity for nonlinear elasticity, and the geometric rigidity of Friesecke, James and Müller, which carries plate theory to 2025. Poisson's derivation ran parallel to Navier's, without using it, and the homogenisation monograph below does not rest on a stated earlier link. No result from 2026 was found.

Hooke and the elastica

In 1660 Robert Hooke found that, for small deformations, the extension is proportional to the force: "ut tensio, sic vis" [1, 2, 3, 4]. He published it as an anagram in 1676 and printed the solution in 1678; the 1678 text gives "ceiiinosssttuu, id est, Vt tensio sic vis" and says he had found it "about eighteen years" earlier (note 1). In the modern general form, the stiffness tensor that links stress to strain has 81 components; its symmetries reduce these to 21 independent ones (a quick count confirms 21). The law does not rest on an earlier link here.

Jacob Bernoulli posed the problem of the elastica, the shape of a bent elastic strip, in 1691 [5, 2, 6]. He solved the rectangular case in 1692–94, and his final paper (1705) states that curvature is proportional to bending moment. A quick check of a property Euler noticed in 1738, according to Levien: if the largest distance of the rectangular elastica from the line of the load is taken as 1, its extent along the load times its arc length is 0.785398 = π/4. Bernoulli's work rests on Hooke's of 1678, whose linear law he questioned by allowing an arbitrary law of elasticity in 1694 (note 2).

In 1742, in a letter to Euler, Daniel Bernoulli proposed that the elastica is the curve that minimises the bending energy $\int \kappa^2 \, ds$ [7, 2, 5]. This idea does not rest on an earlier link here.

In 1744 Leonhard Euler derived the elastica from Daniel Bernoulli's energy by his variational method, and sorted the curves into nine classes [7, 2, 5, 8]. In 1757 he analysed the buckling of columns, with critical load $P = \pi^2 EI/(KL)^2$ (note 3). Two quick checks: for $-y'' = \lambda y$ on an interval of length $L = 2$ with fixed ends, the first eigenvalue is $\lambda_1 = 2.467401 = (\pi/L)^2$; and a pinned steel rod ($E = 200$ GPa) of diameter 10 mm and length 1 m buckles at 968.9 N. Euler's work generalises Jacob Bernoulli's of 1691–1705 and uses Daniel Bernoulli's energy of 1742.

Plates and the field equations

In 1815–16 Sophie Germain won the prize of the Paris Academy with a memoir on vibrating plates [9, 2, 10, 11]. She started from the hypothesis that the elastic force is proportional to the mean curvature, which leads to the fourth-order, biharmonic plate

equation. Her first entry of 1811 had a wrong equation; Lagrange, as a judge, derived the correct one from her hypothesis. MacTutor dates the prize 1815, Holmes January 1816 (note 4).

On 14 May 1821 Claude-Louis Navier read a memoir giving the general equations of an isotropic elastic solid, with a single elastic constant [12, 13]. He derived them from central forces between molecules, linearised in the displacements. This does not rest on an earlier link here.

On 30 September 1822 Augustin-Louis Cauchy announced the concept of stress [12, 2, 14, 11]. The traction on any plane through a point is linear in the plane's normal, T(n) = σn, which he proved by the tetrahedron argument. The isotropic law with two constants came in the published theory, in the *Exercices* of 1828 (Darrigol, Love). A quick check: the announcement came 504 days after Navier's memoir. Navier's work influenced Cauchy: Darrigol says that Navier's appeal to two kinds of restoring forces triggered Cauchy's main inspiration. Holmes names Germain, Fourier and Navier as influences, and Azadi quotes Cauchy crediting Fresnel.

On 14 April 1828 Siméon Denis Poisson read a memoir with a molecular theory giving equations identical with Navier's [13, 12]. It is a parallel derivation: no source says that he used Navier's work.

In 1850 Gustav Kirchhoff gave a plate theory in which fibres normal to the middle surface stay normal after bending [2, 15]. Using virtual work and the calculus of variations, he settled the conditions at a free edge, where the twisting moment and the shear force cannot both be prescribed. Love later gave a theory of thin shells (1888). A quick check: the Navier double series for a simply supported square plate under uniform load gives a maximum deflection w_max = 0.004062 qa^4/D, matching the textbook value 0.00406. Kirchhoff's theory rests on Germain's of 1815–16 and repairs its boundary conditions.

In 1855, in his memoir on torsion, Adhémar Barré de Saint-Venant stated that self-equilibrated loads at the end of a body have only a local effect [16, 17, 18]. A quick check of the decay for a strip: the first Papkovich–Fadle root, z = 2.10620 + 1.12536i, gives decay like e^(−2.106x/h) with h the half-width, a factor of 0.0148 at a distance of two half-widths. The principle does not rest on an earlier link here.

In 1906 Max Born wrote his PhD thesis on the stability of the elastica in the plane and in space [5, 19, 20]. It contains the first accurate plots of the curves, compared with photographs taken with his own apparatus. It rests on Euler's 1744 curves, which it plotted.

Linear elasticity: Korn, Friedrichs and Toupin

Between 1906 and 1909 Arthur Korn proved the inequality now named after him, in papers of 1906, 1908 and 1909 [21, 22]. The L^2 norm of the symmetric part of the gradient of a displacement controls the full gradient, up to rigid motions. A quick check of why rigid motions must be excluded: a skew-symmetric (infinitesimally rigid) gradient has zero symmetric part. Korn's inequality does not rest on an earlier link here.

In 1947 Kurt Friedrichs gave the first modern proof of Korn's inequalities, in the *Annals*; he had treated the two-dimensional case in 1937 [21, 22, 23]. The later standard proof uses a lemma of J. L. Lions. Friedrichs's work came 1947 − 1906

= 41 years after Korn's first paper. It replaces Korn's hard-to-read proofs. Friedrichs said he could not verify Korn's proof of the second case, without boundary conditions, and Nitsche also called it doubtful, according to Durán and Muschietti [23] (note 5).

In 1965 Richard Toupin proved that in an elastic cylinder loaded at one end by a self-equilibrated traction, the strain energy stored beyond a distance z decays exponentially in z (*Archive for Rational Mechanics and Analysis* 18) [16, 24]. This makes Saint-Venant's principle precise, 1965 − 1855 = 110 years after it was stated. Zhao (2011) disputes that it is a full proof of the principle (note 6).

Nonlinear elasticity: quasiconvexity, polyconvexity, microstructure

In 1952 Charles Morrey introduced quasiconvexity (*Pacific Journal of Mathematics* 2, 25–53) [25, 26, 27]. With smoothness and growth conditions it yields minimisers of energies $\int W(\nabla u)$, and it is necessary for them. Its equivalence with weak lower semicontinuity under growth conditions is due to Acerbi and Fusco, according to Wikipedia. Morrey's work does not rest on an earlier link here.

In 1976–77 John Ball proved that minimisers exist in nonlinear elasticity for polyconvex energies, $W(F) = g(F, \operatorname{cof} F, \det F)$ with g convex, under weaker growth conditions than Morrey's [28, 25, 29]. Polyconvex functions are quasiconvex, so the energy is weakly lower semicontinuous. A quick check of why this matters: $\det(F + t\cdot a\otimes b)$ is affine in t, with zero second differences, yet det itself is not convex (in two dimensions diag(2, 0) and diag(0, 2) both have determinant 0, but their average, the identity, has determinant 1). The work rests on Morrey's of 1952 (note 7).

In 1987 Ball and Richard James described fine-phase mixtures as minimising sequences, explaining microstructure in crystals as a failure of quasiconvexity (*ARMA* 100, 13–52) [28, 30, 29]. It uses Morrey's 1952 concept negatively: the minimising sequences form because quasiconvexity fails. The LMS citation says so; the paper's own introduction does not name Morrey or quasiconvexity.

In 1991 or 1992 Olga Oleinik, A. S. Shamaev and G. A. Yosifian published a monograph on homogenisation problems for the elasticity system (Studies in Mathematics and its Applications 26, North-Holland) [31, 32]. No source says which earlier result in this chain it uses (note 8).

Geometric rigidity and plate theory

In 2002 Gero Friesecke, Richard James and Stefan Müller proved geometric rigidity: $\|\nabla v - R\| \le C\|\operatorname{dist}(\nabla v, SO(n))\|$ in L^2, for a single rotation R [33, 34, 35, 36, 37]. As a consequence, nonlinear Kirchhoff plate theory is the Γ-limit of three-dimensional elasticity; the journal paper is in *Communications on Pure and Applied Mathematics* 55. A quick check of the scale: for $F = R(I + 10^{-3}N)$ with N a random normal matrix, dist(F, SO(3)) came out as 1.9×10^{-3} in one run, of order 10^{-3} (note 9).

The proof of the estimate on a cube uses Korn's inequality, according to Lewicka's notes; arXiv 2305.00740 calls the estimate a nonlinear Korn inequality [36, 37]. Both sources say it makes the Liouville–Reshetnyak rigidity theorem quantitative. Kirchhoff's 1850 theory is its target; only the same authors state this link.

In 2006 Friesecke, James and Müller derived a hierarchy of plate models, membrane, Kirchhoff, von Kármán and linear, by Γ-convergence according to how the energy

scales with the thickness (*ARMA* 180(2), 183–236) [38, 39]. Their 2002 rigidity estimate is the key ingredient.

In 2024–25 Manuel Friedrich, Leonard Kreutz and Konstantinos Zemas proved rigidity in variable domains, with a constant that depends only on a curvature integral of the boundary [40, 41]. With it they derived linearised free-surface models by Γ-convergence (*Annales de l'Institut Henri Poincaré C* 42, online September 2024, issue 2025). The work generalises the 2002 rigidity estimate.

In 2025 Tolotti derived the full plate hierarchy for multi-well energies, which model solid–solid phase transitions [38, 42]. An interface penalty restores compactness where rigidity fails. The paper appeared on arXiv in January 2025 and in the *Journal of Nonlinear Science* 35, article 77, in 2025, 2025 − 2002 = 23 years after the rigidity estimate. It extends the 2006 hierarchy of Friesecke, James and Müller, according to the paper itself.

Notes

1. Hooke's 1660, 1676 and 1678 are three events: discovery, anagram and publication. All sources agree on 1678 as the first printing, which MacTutor names as the lecture *Of Spring*. One source renders the solved anagram as "Ut pondus sic tensio"; the 1678 text and the AIP give "Vt tensio sic vis".
2. That Jacob Bernoulli also tested Hooke's law experimentally is stated only by Levien. His 1694 solution and 1705 final statement are compatible.
3. Euler's buckling is dated 1744 by Wikipedia but 1757 by Britannica and the Euler Archive (E238, "Sur la force des colonnes", written 1757, printed 1759 in *Mém. Berlin* 13) [43]. The 1744 work contains the elastica, the 1757 work the column analysis.
4. Germain's prize is 1815 on MacTutor and 1816 in the *Dictionary of Scientific Biography* (Encyclopedia.com); no primary source was consulted.
5. Horgan's 1970 thesis presents Friedrichs's proof as a fuller formulation rather than a correction [44]. Durán and Muschietti say "it is not clear whether the original proof of Korn is correct" for the second case, and that the first correct proofs are likely Friedrichs's, under restrictive assumptions on the domain [23].
6. Zhao (2011) argues that Toupin's theorem is not an exact formulation of Saint-Venant's principle; this is a minority view.
7. Ball's paper is in the December 1976 issue (*ARMA* 63(4), 337–403, per Heriot-Watt's research portal [45]) but is usually cited as 1977 (LMS, Molchanova and Vodop'yanov).
8. The Oleinik–Shamaev–Yosifian monograph is dated 1992 by Elsevier and 1991 by Wikipedia.
9. The value depends on the random matrix drawn; earlier computations gave 1.71 × 10^{-3}. Navier → Cauchy rests on Darrigol and arXiv 2303.13615; Foce (1995) could not be consulted. Not covered in this chain: Green's 21-constant theory (no date in the source), Šverák's 1992 work and Morrey's two-dimensional problem.

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Fluid mechanics

The chain of fluid mechanics starts with Archimedes' hydrostatics, written late in his life before 212 BC. The equations of viscous flow were derived five times from analogies with elasticity, the last time by Stokes in 1845. Stokes' law for a falling sphere (1851) leads on to Reynolds (1883) and Kolmogorov (1941). From the Euler equations four further lines branch off: drag and lift (d'Alembert, Kutta and Zhukovsky, Prandtl), vortex dynamics (Helmholtz, Kelvin, Arnold), weak solutions (Onsager's 1/3 conjecture, proved in 1994 and 2018, then non-uniqueness for Navier–Stokes) and singularities (Elgindi, Chen and Hou, and the forced blow-ups of September 2026).

The Navier–Stokes problem is still listed as open. OpenAI's Lean-checked proof of forced blow-up, announced on 8 September 2026, has not been independently verified, and the Clay Institute is reviewing it. Three pairs of results in this chapter were independent of each other: Kolmogorov and Onsager, Elgindi and Chen–Hou, and Alpöge–Buckmaster and the OpenAI claim. In all, the chapter follows 22 links from Archimedes to the OpenAI claim. Euler's equations, Navier's work of 1822 and Leray's existence theorem are treated in the chapter on differential equations and dynamics.

From floating bodies to the viscous equations

Before his death in 212 BC Archimedes proved, in Book I of *On Floating Bodies*, the principle that bears his name [1, 2, 3]. In Book II he proved results on the stability of floating paraboloids. Both books argue from a single postulate on fluid pressure together with geometric propositions. Nothing earlier in the chain is needed.

In 1752 d'Alembert showed that a body in steady inviscid potential flow feels no drag at all [4, 5]. The argument uses the symmetry of the pressure field around the body; he stated it in 1752 and established it in 1768 for bodies symmetric front to back (note 1). Euler had come close in 1745. A quick check: for a cylinder in potential flow the computed net force is of order 10^{-16}, which is round-off, so it is zero. No source says that d'Alembert's argument uses Euler's equations of ideal fluids. Grimberg, Pauls and Frisch trace it instead to Euler's 1745 work on flow fillets, so the link is one of influence rather than a proof resting on a proof [5].

In 1845 Stokes derived the equations of viscous flow from a linear law relating stress to rate of strain, and justified the same form for elastic solids [6, 7]. He then found that Navier, Poisson and Saint-Venant had the equations already. This was a rediscovery, not a step built on Navier: Navier's equation of 1822 came from his reasoning about elasticity (see the chapter on elasticity and solids). A quick check: Stokes wrote 23 years after Navier's fluid memoir of 18 March 1822 (note 5).

In 1851 Stokes found the drag on a small sphere, $F = 6\pi\mu R v$, in his memoir on pendulums [8, 6]. The law solves the slow-flow, low-Reynolds-number limit of the Navier–Stokes equations, and so builds on his 1845 equations (note 10). A quick check: a sand grain of radius 0.1 mm falls through water at 0.03597 m/s, but its Reynolds number is then 7.19, outside the law's range of validity.

Vortices and circulation

In 1858 Helmholtz proved three theorems on vortices in an inviscid fluid [9, 10, 11]. Vortex lines move with the fluid; vortex tubes either close on themselves or end on boundaries; and the strength of a tube is conserved. The proof rests on Euler's equations. Today these theorems are usually proved by way of Kelvin's theorem.

Kelvin proposed vortex atoms in 1867 and proved the circulation theorem in 1869 (note 3) [12, 13, 14, 15]. The theorem says that DΓ/Dt = 0 for a barotropic ideal fluid: the circulation around a loop moving with the fluid does not change. The proof takes the material derivative of the loop integral. A quick check: a rigid rotation with Ω = 0.7 gives a circulation of 4.398230 around the unit circle, which is 2πΩ. The vortex atoms rest on Helmholtz's 1858 work, by way of Tait's smoke-ring experiments that tested Helmholtz's prediction [12].

In 1966 Arnold showed that the motions of an ideal fluid are geodesics of the kinetic-energy metric on SDiff(D), the group of volume-preserving diffeomorphisms of the domain [30, 31]. This generalises Euler's rigid body, whose motions are geodesics on SO(3). Where the curvature is negative in many directions, the flow is unstable. The paper's abstract says it extends Euler's theorems on geodesics of SO(3) to arbitrary Lie groups, so it rests on Euler's rigid body (see the chapter on mechanics) (note 11).

Turbulence, lift and the boundary layer

In 1883 Reynolds made his dye-streak experiments in pipes [16, 17, 18]. They showed that one dimensionless number governs the transition from laminar to turbulent flow. The Reynolds-averaged equations followed in 1895 (note 4). A quick check: water in a pipe 2 cm across reaches Re = 2300 at a speed of 0.1150 m/s. Reynolds built on Stokes' paper of 1851, read in 1850, which first noted this similarity [18].

Kutta in his habilitation thesis of 1902, and Zhukovsky in papers of 1906, found independently that the lift per unit span is ρVΓ, with the sign fixed by convention [19, 20, 21]. The law holds for any closed body in a uniform two-dimensional inviscid stream with circulation Γ. A quick check: a cylinder with Γ = 2.5 gives a lift of −2.500000, which is −ρUΓ, and a drag of order 10^{-16}. Neither source names d'Alembert, so this law was found independently of his paradox.

In August 1904, at the Heidelberg congress, Prandtl introduced the boundary layer [22, 4, 23]. The limit of vanishing viscosity is singular: a thin layer with friction forms at the walls. The flow splits into an outer inviscid region and this inner layer. The method resolves d'Alembert's paradox of 1752, and Saint-Venant had already given the diagnosis in 1846 (note 2).

In 1941 Kolmogorov proposed his theory of local isotropy at high Reynolds number, known as K41 [24, 25, 26]. It gives the second-order structure function $S_2(r) \sim (\varepsilon r)^{(2/3)}$ and the exact third-order law $S_3 = -(4/5)\varepsilon r$, from a closure of the Kármán–Howarth equation. A quick check: the slope 2/3 in r corresponds to the −5/3 law of the energy spectrum. The theory rests on Richardson's picture of the energy cascade and on the Kármán–Howarth equation.

Weak solutions and Onsager’s conjecture

In 1949 Onsager conjectured that solutions of the Euler equations with Hölder exponent above 1/3 conserve energy, while those below 1/3 may dissipate it [27, 28, 29]. The argument was a heuristic estimate of the energy flux. Onsager did not build on Kolmogorov’s theory by proof: he was influenced by it and refined it with a non-statistical argument about single solutions (note 10).

In 1994 the rigid half of the conjecture was proved: energy is conserved for exponents above 1/3 [28, 27, 32]. Eyink proved it in part and Constantin, E and Titi in full, using a commutator estimate. Their paper appeared in Communications in Mathematical Physics 165, 207–209, in October 1994. This proves the first half of Onsager’s 1949 conjecture.

The flexible half took longer. Isett, in a paper posted on arXiv in 2016 and published in the Annals in 2018, built weak Euler solutions in C^α for every α below 1/3 that are compactly supported in time [27, 28, 33]. The method is convex integration with gluing and Mikado flows. It rests on Onsager’s conjecture, which it proves, and on Nash’s convex integration as developed by De Lellis and Székelyhidi. A quick check: the proof came 69 years after Onsager’s conjecture.

In 2017 Buckmaster, De Lellis, Székelyhidi and Vicol built strictly dissipative solutions in C^β, for β below 1/3, with any smooth positive energy profile [28, 34, 35]. They also used convex integration and gluing; the paper was posted as arXiv 1701.08678 and published in Communications on Pure and Applied Mathematics 72(2), 229–274, in 2019. It builds on Isett’s work, whose solutions were not strictly dissipative; the two groups of authors overlap.

In 2019 Buckmaster and Vicol showed that the three-dimensional Navier–Stokes equations have infinitely many finite-energy weak solutions that are not Leray–Hopf solutions [36, 37]. The method is an intermittent form of convex integration, first posted on arXiv in 2017. Their own survey says the proof builds on the Nash-type convex integration scheme of De Lellis and Székelyhidi for the Euler equations, with intermittent building blocks added [38]. The question of uniqueness goes back to Leray’s existence theorem of 1934, which their abstract cites.

Singularities and blow-up

In 2021 Elgindi proved that C^(1,α) solutions of the three-dimensional Euler equations on $\mathbb{R}^3$ can develop a singularity in finite time, even for some of the simplest flows [39, 40]. The paper was posted on arXiv in 2019 and published in the Annals 194(3), 647–727. The solutions he constructs are exactly self-similar, and axially symmetric without swirl [39]. The result uses the local theory of Lichtenstein and Gunther from the 1920s, which the abstract cites.

In 2022 Albritton, Brué and Colombo proved that Leray solutions of the forced three-dimensional Navier–Stokes equations are not unique [41, 42]. The second solution lies on the unstable manifold of a self-similar vortex ring; the paper was posted on arXiv in December 2021. It builds on Leray’s work of 1934, on Vishik’s 2018 work and on Jia and Šverák’s of 2015, all named in the paper. A quick check: the result came 88 years after Leray.

Between 2022 and 2025 Chen and Hou proved stable, nearly self-similar blow-up for

the two-dimensional Boussinesq equations and for three-dimensional axisymmetric Euler, with smooth data and a boundary [43, 44, 45, 46, 47]. This was the first rigorous justification of the Hou–Luo scenario. Part I uses weighted estimates; Part II uses computer-assisted rigorous numerics and appeared in SIAM Multiscale Modeling and Simulation 23(1), 25–130, in 2025. A summary appeared in PNAS 122(27) in July 2025. Part I is still a preprint, at version 4 of 16 August 2026 (checked on 28 October 2026). A quick check: from version 1 (13 October 2022) to version 4 took 1403 days. The work rests on the Hou–Luo numerical scenario, which it proves; the arXiv abstract cites the numerical evidence. Chen and Hou did not build on Elgindi, according to one source; the two lines are parallel.

In 2025 Wang and co-authors found new families of unstable self-similar singularities numerically, for the incompressible porous medium (IPM) equation and for three-dimensional Euler with a boundary (note 7) [48, 49]. Their precision reaches nearly double precision, enough for later computer-assisted proofs, but no proof is claimed. The paper is a preprint (checked on 28 October 2026). It cites Chen and Hou's computer-assisted proof and the Luo–Hou evidence as background, so the link is one of influence rather than a proof built on theirs [58].

In September 2026 Alpöge and Buckmaster posted a proof of finite-time blow-up for the two-dimensional Boussinesq equations on $\mathbb{R}^2$ [50, 51]. The forcing is smooth and supported in a ball, and the initial velocity is zero. A companion result, with Coiculescu, covers IPM on the torus T^2 (arXiv 2609.16470). The paper's statement on AI use mentions earlier write-ups for three-dimensional Euler and for hypodissipative Navier–Stokes. It says that the first blow-up solution was produced on 15 August and verified in Lean on 22 August 2026. Both are preprints (checked on 28 October 2026). The work rests on the multiscale construction of Córdoba and Martínez-Zoroa (note 6).

On 8 September 2026 OpenAI announced a Lean-checked proof of finite-time blow-up for the forced three-dimensional Navier–Stokes equations, with smooth data at rest, a smooth force and finite energy [52, 53, 54, 55, 56]. No refereed check had been found as of 29 October 2026. Duraiswami's arXiv 2609.17642, posted on 15 September 2026, analyses the swirl core of the construction but does not check the proof. The Clay statement of 11 September 2026 says the problem has "apparently been settled" and that its own process is "deliberately unhurried"; the Clay problem page still lists it as active (both checked on 29 October 2026). A quick check: 50 days passed from the announcement to 28 October 2026.

The claim rests on Fefferman's official statement of the Clay problem. Parts (A) and (B) set the force to zero, but parts (C) and (D), the breakdown statements, ask for a smooth force f satisfying decay conditions, so a forced blow-up answers them [57]. The OpenAI claim and the Alpöge–Buckmaster work were independent of each other; their relation is a dispute over priority, not a proof resting on another [55, 54] (note 9).

Notes

1. D'Alembert's paradox is dated 1752 by Wikipedia and 1768 by Grimberg, Pauls and Frisch; the primary text does not settle it [4, 5].
2. Prandtl (1904) is credited with resolving the paradox by Wikipedia and by Narasimha in *Nature*, through the method of matched asymptotics that grew

from his approach [4, 23]. Grimberg, Pauls and Frisch credit Saint-Venant (1846) with the diagnosis [5]; Wikipedia dates his viscous model to 1847. The two credits are compatible.

3. Kelvin's circulation theorem is dated 1869 on one Wikipedia page and 1867 on another [13].
4. MacTutor dates Reynolds' turbulence work to 1889, while Rott dates the Reynolds-averaged equations and the explicit critical Reynolds number to 1895 [16, 18]. Rott also dates Stokes' similarity remark to 1850; the paper was read in 1850 and printed in 1851.
5. The Navier–Stokes equations were derived five times: by Navier (1822), Cauchy, Poisson (1829), Saint-Venant (1843) and Stokes (1845) [6, 7].
6. The Alpöge–Buckmaster posting is dated 7 September by Unite.ai and 8 September by NYU Shanghai RITS [54, 55]. Their forced-Euler preprint was not seen.
7. Physics World adds the Boussinesq equations to the results of Wang and co-authors, while the abstract names only IPM and three-dimensional Euler with a boundary [48, 49].
8. Archimedes is often dated to c. 250 BC. Wikipedia places *On Floating Bodies* late in his life, and he died in 212 or 211 BC [1, 3].
9. The Clay statement of 11 September 2026 does not name OpenAI [52]. OpenAI's own page could not be read, and no OpenAI paper or arXiv number was seen; arXiv 2609.17642 describes the construction but does not verify it [56].
10. Only Wikipedia states that Stokes' law of 1851 solves the slow-flow limit of the 1845 equations; MacTutor does not link the two papers [8, 6]. Onsager's 1949 paper was not consulted directly; the account here follows the Annals, the Buckmaster–Vicol survey and Eyink's 2024 essay [27, 28, 29]. Reynolds' 1883 paper was not consulted directly either.
11. The dependencies of Arnold (1966), Buckmaster–Vicol (2019), Elgindi (2021) and Albritton–Brué–Colombo (2022) are stated only by their own authors [30, 38, 39, 41]. Chen and Hou's Part I has no journal reference yet (arXiv version 4 of 16 August 2026) [43].

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Optics and electromagnetism

The chain of optics starts with Hero's proof of the law of reflection from a shortest path, in the first century AD. It runs through the sine law of refraction, Fermat's principle of least time (1662), Huygens' wavelets and Fresnel to the integral theorem of Helmholtz and Kirchhoff (1859 and 1882). Maxwell's papers of 1861–62 and 1865 and his *Treatise* of 1873 lead to Heaviside's four equations (1884) and to Lorentz invariance. On Maxwell's equations rest Mie's solution for the sphere (1908) and the cloaks of transformation optics (2006). Through Sommerfeld's radiation condition (1912) and Rellich's lemma (1943) they also carry the uniqueness and sampling theory of inverse scattering.

Two pairs of results were independent of each other: Huygens' construction and Fermat's principle, and the 2003 counterexamples to Calderón's problem and the 2006 cloaks, which share one singular map. For 2024–26 only preprints and one short note were found. The chapter follows 27 links from Hero to Godin and Vainberg.

Reflection, refraction and least time

In the first century AD Hero derived the law of reflection in his *Catoptrica* from the principle that light takes the shortest total path [1, 2, 3]. Reflect one endpoint in the mirror; the straight line to the other endpoint is the shortest path, so the angles of incidence and reflection are equal. A quick check: for A = (0, 1) and B = (3, 2) above a mirror on the x-axis, the shortest path makes angles of 45.00000° going in and coming out. Nothing earlier in the chain is needed (note 4).

In 984 Ibn Sahl used the sine law of refraction to design lenses free of aberration, in his treatise *On Burning Mirrors and Lenses* [4, 5]. A quick check: this was 637 years before Snell. Whether his use of the law counts as a discovery of it is disputed (note 9).

The sine law was then found again. Harriot had it in 1602 and Snell in 1621, but neither published it; Snell's result became known when Huygens printed it in his *Dioptrica* of 1703 [6, 7, 5]. Descartes derived the law in 1637 from a "velocity" whose component along the surface is conserved. These were rediscoveries, not steps built on Ibn Sahl. According to Hentschel's reconstruction, as reported by MacTutor, Snell took a special interest in Ibn al-Haytham's *experimentum elegans* [7]. Whether Descartes found the law independently is unclear [5] (note 12).

In 1662 Fermat derived the law of refraction from the principle of least time, and rejected Descartes' argument [6, 8, 9, 3]. The usual date is a letter of 1 January 1662 to La Chambre (note 12). A quick check: take A = (0, 1) in a medium with speed $v_1 = 1$ and B = (1, −1) in one with $v_2 = 1/1.5$, with the interface at y = 0. Minimising the travel time over the crossing point gives $\sin\theta_1/\sin\theta_2 = 1.49999996$, the expected ratio of 1.5. Fermat's derivation stands as a rival to Descartes' of 1637, and Hero's shortest path was a precursor that influenced it without being used in its proof.

Waves, wavelets and rays

Huygens drafted his wave theory in 1678 and printed it in 1690 [10, 11, 1]. Every point of a wavefront emits secondary wavelets, and the envelope of these wavelets gives the next wavefront; from this he obtained the sine law. His construction implies Fermat's principle of least time, but Huygens did not notice this, so the two results stand side by side without one resting on the other. Only Wikipedia states this [1].

In 1818 Fresnel combined Huygens' wavelets with interference [10, 11]. This explained both straight-line propagation and diffraction, but needed an assumed obliquity factor. The work rests on Huygens' principle of 1690.

From 1826 to 1832 Hamilton developed the characteristic function for systems of rays [12, 13, 14]. He presented his memoir *Theory of Systems of Rays* in 1826 and the third supplement in 1832. The theory predicted conical refraction, which he presented on 22 October 1832; Lloyd observed it on 14 December. Hamilton carried the method into dynamics in 1834 (see the chapter on mechanics).

Helmholtz in 1859 and Kirchhoff in 1882 gave Huygens' principle a rigorous integral form, stationary in Helmholtz's case and time-dependent in Kirchhoff's [10, 11]. Their work derives the assumptions that Fresnel had made ad hoc, and so rests on Fresnel's 1818 theory. A quick check: Kirchhoff's form came 1882 − 1818 = 64 years after Fresnel.

Maxwell's equations and their forms

In 1855–56 Maxwell wrote "On Faraday's Lines of Force", which gave Faraday's lines a mathematical form through an analogy with fluid flow [15, 16]. It starts a new line and rests on no earlier link in this chain.

In 1861–62 Maxwell's "On Physical Lines of Force" added the displacement current to Ampère's equation [17, 15]. From a model of molecular vortices he concluded that light is a transverse wave in the same medium as electric and magnetic effects. A quick check: with the CODATA 2018 values $\mu_0 = 1.25663706212 \times 10^{-6}$ N/A² and $\varepsilon_0 = 8.8541878128 \times 10^{-12}$ F/m, $c = 1/\sqrt{(\mu_0\varepsilon_0)} = 299\,792\,458.0$ m/s. The paper builds on his work of 1855–56.

In 1865 "A Dynamical Theory of the Electromagnetic Field" gave the first complete system [18, 15]. It had 20 equations in 20 variables, arranged in groups (A) to (H), and no longer used the vortex model. Part VI gives the electromagnetic theory of light. It rests on the paper of 1861–62.

In 1873 Maxwell's *Treatise*, Article 619, restated the equations in 12 entries, partly in quaternion form [19, 16, 15]. Wikipedia speaks of "twelve general equations", including the eight of 1865. A quick check: 5 lettered and 7 unlettered entries make 12. The *Treatise* rests on the paper of 1865.

In 1884 Heaviside reduced the system to four vector equations in two variables, E and H, by eliminating the potentials; Gibbs also used vector methods [15, 20, 21, 22]. Heaviside worked this out in 1884 and printed it in *The Electrician* from 1885. A quick check: this was 11 years after the *Treatise*. It rests on Maxwell's work of 1865 and 1873 (note 1).

Between 1895 and 1904 Larmor and Lorentz showed that the Lorentz transformation

leaves Maxwell's equations unchanged, and in 1905 Einstein made this the basis of special relativity [15, 23]. A quick check: Einstein's paper came 40 years after Maxwell's of 1865. The work rests on Maxwell's equations in general, whose invariance Larmor and Lorentz derived; no source says that it used Heaviside's form of 1884 in particular (note 8).

Solutions of Maxwell's equations

In 1908 Mie solved exactly the scattering of a plane wave by a homogeneous sphere [24, 25]. He expanded the solution of Maxwell's equations in vector spherical harmonics; Lorenz had found the solution independently in 1890 (note 5). A quick check: for refractive index m = 1.5, with the Bohren–Huffman series summed to x + 4x^(1/3) + 2 terms, the scattering efficiency at size parameter x = 0.1 is Q_sca = 2.3084 × 10⁻⁵, against Rayleigh's approximation of 2.3068 × 10⁻⁵. The extinction efficiency tends to 2: it is 2.882, 2.094 and 2.014 at x = 10, 100 and 1000. The work rests on Maxwell's equations.

In 2006 Pendry, Schurig and Smith, and independently Leonhardt, proposed cloaks based on transformation optics [36, 34, 37]. Because Maxwell's equations keep their form under a change of coordinates, a singular coordinate change gives the permittivity ε and permeability μ of a cloak. Leonhardt's two-dimensional cloak with a graded index and a microwave demonstration by Schurig and co-workers followed. A quick check: with the textbook formulas, a cylindrical cloak with inner radius a = 1 and outer radius b = 2 has ε_θ = 1001 at r = 1.001; it becomes singular at the inner radius. The work rests on the form invariance of Maxwell's equations. It does not rest on Greenleaf, Lassas and Uhlmann (below): the two use the same map, but no dependency is stated.

In 2026 Godin and Vainberg recast boundary and transmission problems for the time-harmonic Maxwell equations as elliptic boundary-value problems, by adding two extra scalar unknowns [40]. The work is a preprint (checked on 28 October 2026). Its abstract names no earlier result it builds on, so it stands without a link to the rest of the chain.

Radiation, uniqueness and inverse scattering

In 1912 Sommerfeld stated the radiation condition r(∂u/∂r − iku) → 0 [26, 27]. It selects the unique outgoing solution of the Helmholtz equation. A quick check with k = 2: the residual is 1/r for the outgoing wave e^(ikr)/r, giving 0.1, 0.01 and 0.001 at r = 10, 100 and 1000. For the incoming wave e^(−ikr)/r it tends to 2k = 4, giving 4.00125, 4.000012 and 4.0000001. It starts a new line in this chain.

In 1943 Rellich, and independently Vekua, proved that a solution of the Helmholtz equation outside a ball vanishes identically if its L^2 mass on large spheres tends to zero (note 6) [26, 27]. The proof expands the solution in spherical harmonics and solves the radial ordinary differential equation. A quick check: the outgoing wave e^(ikr)/r has mass 4π = 12.566371 on every sphere, so it does not tend to zero. Sommerfeld's condition enters only as a definition: Colton and Kress state the lemma for radiating solutions. It is uniqueness under the radiation condition that is proved through the lemma, not the reverse.

From the 1950s to 1969 Müller, with Vekua and Weyl, proved existence for scattering

by obstacles using boundary integral equations, which can be solved by the Riesz–Fredholm theory [27, 28, 29]. Colton and Kress also credit him with a unique continuation principle. His monograph *Foundations of the Mathematical Theory of Electromagnetic Waves* appeared with Springer in 1969 as Grundlehren 155. According to Colton and Kress, this work rests on Rellich's lemma of 1943; Kress's survey ties uniqueness only to the radiation condition [27, 28] (note 11).

Around 1960 Schiffer proved that a sound-soft obstacle is determined by its far fields for all directions of incidence [27, 28, 30]. The result was never published by him; it appears in the book of Lax and Phillips. The proof uses Rellich's lemma to make the scattered waves coincide outside the obstacles. Colton and Kress state this, and Stefanov and Uhlmann call it "a well-known argument based on Rellich's lemma".

In 1988 Nachman, Novikov and Ramm proved that the refractive index is determined by far-field data, using complex geometric optics solutions [27, 31] (note 10). The proof uses Rellich's lemma to show that the fields are equal outside the support of the medium, as Colton and Kress state; Hu and Liu use it in the same argument. According to Colton and Kress, it also uses Müller's unique continuation principle [27].

In 1990 Isakov proved uniqueness for obstacles by means of singular solutions that approach the boundary, and in 1993 Kirsch and Kress simplified his proofs [27, 28, 30]. The simplification rests on Isakov's work.

In 1992 Colton and Kress completed their monograph *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, printed in 1993 [27, 32]. Springer gives the copyright year as 1992. The book collects the results above and is not itself a new link.

In 1996 Colton and Kirsch introduced the linear sampling method in *Inverse Problems* 12, and in 1998 Kirsch introduced the factorization method [27, 33] (note 7). According to Colton and Kress, linear sampling grew out of the dual space method of Colton and Monk [27]. The factorization method is a refinement of linear sampling that removes its theoretical gaps [41]; according to Colton and Kress, it also gives a new proof of Schiffer's theorem [27].

Invisibility and single measurements

In 2003 Greenleaf, Lassas and Uhlmann found degenerate conductivities that cannot be detected [34, 35]. Blowing a point up to a ball hides the interior from the Dirichlet-to-Neumann map. This gives counterexamples to uniqueness in Calderón's problem (MSC 35R30), so the work rests on that problem as a counterexample to it; the account comes from the same group of authors.

In 2024 Liu solved Schiffer's problem of uniqueness from a single measurement "almost surely", in a probabilistic sense [38, 39]. The note appeared in *Communications in Analysis and Computation* 2(3), 294–299, in September 2024. It rests on Schiffer's conjecture.

Notes

1. When Maxwell's equations first appeared is settled from the primary texts: their content in 1861–62, the complete system in 1865, the *Treatise* in 1873, and the four vector equations worked out in 1884 and printed from 1885 [21, 22].

MacTutor's phrase "fully developed form" for 1873 judges completeness, not first appearance [16].

2. Huygens' wave theory was drafted in 1678 and printed in 1690. Wikipedia's page on Snell's law dates the *Traité* to 1678, while MacTutor has Huygens publishing Snell's law in the *Dioptrica* (1703); these are two different works, so there is no conflict [6, 7].
3. Hertz's experiments are dated both 1887–88 and the 1890s on Wikipedia, which is inconsistent; Hertz is not a link in this chain.
4. MacTutor dates Hero to c. AD 10–75, with Neugebauer's eclipse dating of AD 62; older proposals of 150 BC or AD 250 are rejected [2].
5. Wikipedia calls the sphere solution Mie's, while Wriedt also credits Clebsch (1863), Lorenz (1890) and Debye (1909); both agree that Lorenz was earlier and independent [24, 25]. Mie's paper is *Annalen der Physik* 25 (IV), 377–445 (1908), which is volume 330 in the continuous numbering.
6. Rellich's lemma is credited to Rellich alone by the TU Dortmund notes and to Rellich and Vekua by Colton and Kress [26, 27].
7. Linear sampling is Colton and Kirsch, *Inverse Problems* 12 (1996), and the factorization paper is Kirsch (1998), as given by Colton, Giebermann and Monk [33]; the list in Colton and Kress has entries dated 1989 and 1999 [27].
8. MacTutor dates Lorentz's stages to 1895 and 1904, Wikipedia to 1899 and 1904, with Larmor in 1897 [23, 15].
9. Rashed, via Wikipedia, credits Ibn Sahl with the sine law, while Young argues that his use of it was not a true discovery of the law, since no measurements are given [4, 5].
10. Hu and Liu credit the classical uniqueness result for the refractive index to Sylvester and Uhlmann rather than to Nachman, Novikov and Ramm; this is not settled [31].
11. Müller's German original was not seen. His unique continuation step and its use by Nachman, Novikov and Ramm, the origin of linear sampling in the Colton–Monk dual space method, and the factorization method's new proof of Schiffer's theorem rest on Colton and Kress alone. The confirmations of Müller and of Isakov, Kirsch and Kress lean on Kress's survey, written by a co-author of Colton and Kress [27, 28].
12. Snell's interest in Ibn al-Haytham's experiment rests on MacTutor's account of Hentschel's reconstruction, and the exact date of Fermat's letter of 1 January 1662 rests on Wikipedia alone [7, 1]. Godin and Vainberg's preprint has no second source, and no peer-reviewed result of 2025–26 was found [40].
13. The pair Sommerfeld–Rellich is a link by definition rather than by use in a proof, and Lorentz invariance rests on Maxwell's equations in general rather than on Heaviside's form.

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Thermodynamics

Two lines run side by side in this field, with no link between them. Heat transfer starts with Fourier's heat equation and series (1807–1822). It runs through Stefan's free-boundary problem (1889), Duvaut's reduction to an obstacle problem (1973) and Caffarelli's regularity theory (1977) to the theorem of Figalli, Ros-Oton and Serra on the singular set (*Journal of the American Mathematical Society*, 2024).

Thermodynamics proper is a chain of axiomatisations. It runs from Carnot (1824) through Kelvin and Clausius (1848–1865) to Carathéodory (1909) and Lieb and Yngvason (1998–99). Stefan's work was independent of the earlier problem of Lamé and Clapeyron, and no result of 2025–26 was found. The chapter follows 16 links from Fourier to Figalli, Ros-Oton and Serra.

Heat conduction and Fourier series

Fourier treated heat conduction in solids by means of the heat equation and infinite trigonometric series [1, 2]. His memoir was read on 21 December 1807, won a prize in 1811 (note 1), and became the book *Théorie analytique de la chaleur* in 1822. A quick check: for an initial temperature $u_0 = x(\pi - x)$ on $(0, \pi)$, with $u = 0$ at both ends, an explicit finite-difference solution (400 intervals, $\Delta t = 0.4\Delta x^2$) and the Fourier series (odd n up to 2000) agree to within 5.25×10^{-6} at $t = 0.5$. Nothing earlier in the chain is needed.

In 1829 Dirichlet proved the first rigorous theorem on the convergence of Fourier series, in Crelle's journal 4, 157–169, dated January 1829 (note 4) [6, 7, 21]. He found that Cauchy's earlier proof was defective. A quick check: this was 22 years after the memoir of 1807. Dirichlet's paper starts from the series of Fourier's memoir on heat, as the paper itself says [7]. Lagrange and Laplace had objected to Fourier's series in 1808 [1]; a later historian calls Dirichlet's proof "the first satisfactory exposition from the mathematical standpoint" [21].

Free boundaries: from Lamé and Clapeyron to Stefan

In 1831 Lamé and Clapeyron computed the thickness of the solid crust that forms as a liquid globe cools [8, 9, 20]. This is the heat equation with a moving front between solid and liquid. They found that the thickness grows as the square root of time, controlled by the removal of latent heat. Their problem uses Fourier's theory of heat conduction, which Ågren names as its background, although no source says they applied his equation directly [20] (note 8).

Around 1860 Neumann found a similarity solution for solidification [8, 11] (note 6). The front moves as $x = 2\lambda\sqrt{t}$, the temperature has an error-function profile, and λ satisfies a transcendental equation. The work survived in his lectures and was published through Weber in 1901. A quick check: solving the one-phase equation λ e^(λ^2) erf $\lambda = \mathrm{St}/\sqrt{\pi}$ by Brent's method gives $\lambda = 0.220016$ at Stefan number St =

0.1 and $\lambda = 0.620063$ at $\mathrm{St} = 1$. The Stefan condition at the front then holds to nine digits.

In 1889 Stefan wrote four papers on free boundaries: on ice in polar seas, on diffusion and on evaporation [8, 9] (note 5). He derived from an energy balance at the interface that the latent heat times the speed of the front equals the jump in heat flux. The ice paper was reprinted in 1891. A quick check: this came 58 years after Lamé and Clapeyron. Stefan uses Fourier's heat equation in each phase, and his problem has Neumann's solution as a special case. Lamé and Clapeyron were a precursor, but no dependency on them is stated.

Weak solutions, obstacles and regularity

In 1960 Oleinik, and in 1961 Kamenomostskaya, proved existence and uniqueness of a generalised (weak) solution of the multidimensional Stefan problem [11, 9]. According to Wikipedia, Oleinik built on her own earlier work. A quick check: $1960 - 1889 = 71$ years after Stefan. The work rests on Stefan's problem of 1889, which it solves in a weak sense.

In 1973 Duvaut, in a note in the *Comptes rendus* of the Paris Academy, turned the one-phase Stefan problem into a parabolic obstacle problem [14, 9]. The substitution $u = \int \theta \, d\tau$, integrating the temperature over time, does this; the free boundary becomes the edge of the contact set. A quick check: on Neumann's solution with $\mathrm{St} = 1$ ($\lambda = 0.620063$), at $x = 0.5$ the function u stays at 0 until $t = (x/2\lambda)^2 = 0.1626$, and is positive after that. The reduction rests on Stefan's problem.

In 1977 Caffarelli proved, in *Acta Mathematica* 139, that the free boundaries of the obstacle problem are regular near regular points [14, 11]. Through Duvaut's reduction the same holds for Stefan free boundaries. The method uses blow-ups and a classification of global solutions. A quick check: this was 4 years after Duvaut. The result rests on Duvaut's reduction of 1973.

In 2024 Figalli, Ros-Oton and Serra bounded the singular set of the Stefan problem [11, 14, 18, 19]. In $\mathbb{R}^3$ the free boundary is smooth for almost every time. The paper was posted as arXiv 2103.13379 on 24 March 2021 and appeared in the *Journal of the American Mathematical Society* in 2024 (note 7). A quick check: $2024 - 2021 = 3$ years from arXiv to journal. It rests on Caffarelli's theory of 1977, whose singular points it analyses.

The second law: from Carnot to Clausius

In 1824 Carnot proved that all reversible engines working between two reservoirs have the same, maximal efficiency [3, 4, 5]. The proof is by reductio from the impossibility of perpetual motion, and assumes that caloric (heat) is conserved. A quick check: between the ice and steam points the efficiency is $\eta = 1 - 273.15/373.15 = 0.267989$. This starts the second line.

Between 1848 and 1854 Kelvin built the absolute temperature scale (1848) [3, 4]. In 1851 he made an axiom of the statement that heat does not pass to a hotter body without outside agency. In 1854 he gave $W/Q = 1 - T_2/T_1$ and $\Sigma Q_i/T_i = 0$ for reversible cycles. A quick check: a reversible cycle that takes $Q_1 = 1000$ J at $T_1 = 500$ K and rejects 600 J at $T_2 = 300$ K gives $\Sigma Q/T = 2 - 2 = 0$. For the scale, Kelvin rests on

Carnot's theorem, which he knew through Clapeyron. For the axiom, he rests on a remark of Clausius from 1850.

In 1850 Clausius kept Carnot's theorem but dropped the conservation of heat [3, 4]. He used the equivalence of heat and work found by Joule and Mayer, the first law. He re-derived the theorem from the impossibility of perpetual motion of the second kind. The work rests on Carnot's of 1824, with caloric replaced.

In 1854 Clausius stated that heat cannot of itself pass from a colder to a hotter body (note 2) [3, 10]. By combining cycles with Carnot engines he showed that $\oint dQ/T = 0$ for reversible cycles and $\oint dQ/T \leq 0$ for irreversible ones. A quick check: an engine that takes 1000 J at 500 K and rejects 700 J at 300 K has efficiency 0.3, below Carnot's 0.4, and gives $\Sigma Q/T = -0.3333$ J/K.

In 1865 Clausius defined entropy S through the path-independence of $\int dQ/T$ for reversible paths [3, 10]. He showed that S does not decrease in adiabatic processes, by closing an irreversible path with a reversible return. The work rests on his result of 1854.

Axiomatic thermodynamics

In 1909 Carathéodory took as an axiom that every neighbourhood of a state contains states that cannot be reached adiabatically [10, 12, 13]. Temperature then appears as an integrating factor, in *Mathematische Annalen* 67, 355–386. A quick check: the paper has 32 pages. It rests on Clausius' entropy, which it rederives from axioms: absolute temperature and entropy are defined from the Pfaffian form of quasi-static adiabatic changes (note 9).

In 1998–99 Lieb and Yngvason derived the existence and uniqueness of entropy, and the second law, from postulates on adiabatic accessibility [3, 15, 16, 17]. The comparison principle, that any two states can be compared, is derived from assumptions on pressure and thermal equilibrium rather than assumed. The work was posted on arXiv in August 1997 and appeared in *Physics Reports* 310 in 1999. A quick check: $1999 - 1909 = 90$ years after Carathéodory. It continues Carathéodory's axiomatic line without assuming that all states are comparable, as Yngvason states [16].

Notes

1. MacTutor dates Fourier's prize to 1811 and flags "not 1812", while many accounts say 1812 [1].
2. The Clausius statement is an idea of 1850 and an explicit axiom of 1854 [3].
3. Planck's principle is dated 1926 in Wikipedia's article on the second law, citing Planck 1926, p. 457 [10]. Uffink cites only Planck 1897, 1945 and 1948 [3], and no second source was found, so the 1926 date rests on Wikipedia alone. Planck is not a link in this chain.
4. Dirichlet's paper is dated from the paper itself as Crelle 4 (1829), 157–169, Berlin, January 1829; MacTutor gives 1828 [7, 6].
5. Stefan's work is dated 1889 (four papers), with the 1891 reprint listed by the Encyclopedia of Mathematics; Wikipedia says "around 1890" [9, 11].
6. Neumann's solution is dated "early 1860s" by Wikipedia and "about 1860" by Vuik; these are compatible [11, 8].

7. The volume and pages of the Figalli–Ros-Oton–Serra paper in the *Journal of the American Mathematical Society* were not confirmed from a primary page; the year 2024 rests on CVGMT and Wikipedia [18, 11].
8. That Lamé and Clapeyron used Fourier's equation is an inference from the content of their problem; Ågren names Fourier's work only as background [20]. The primary papers of Oleinik and Kamenomostskaya were not consulted.
9. Duvaut's note of 1973 is cited by the Encyclopedia of Mathematics for the reduction to a variational inequality, beside Ros-Oton [9, 14]. Carathéodory's axiom was read in Delphenich's translation [12]; his theorem on Pfaffian forms and later repairs of his proof were not consulted. Lieb and Yngvason place their work in Carathéodory's line [16]; their article in the *Notices of the AMS* could not be read. Gibbs is not sourced here and so is not a link.

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Quantum theory

Quantum theory as mathematics starts with Planck's radiation law of 1900, derived with Boltzmann's way of counting states. It reaches matrix mechanics and wave mechanics in 1925–26. From there three lines run: operators (Stone–von Neumann, Kato, the stability of matter), measurement and probability (Born, Kennard and Robertson, Gleason, Kochen and Specker) and nonlocality (Einstein–Podolsky–Rosen, Bell, CHSH, Tsirelson).

In 2026 the nonlocality line ends in complexity theory, with MIP* = RE (2020) and MIPco = coRE (2025), while the Yang–Mills mass gap is still open. No link runs from any ancient root into this chain. Kennard and Weyl found the uncertainty bound in parallel, and Lieb and Thirring's proof of the stability of matter was new rather than built on Dyson and Lenard's. The chapter follows 23 links from Planck in 1900 to the open Yang–Mills mass gap.

From Planck's quanta to matrix mechanics

In 1900 Planck derived his radiation law, $B(\nu) = (2h\nu^3/c^2)/(e\hat{}(h\nu/kT) - 1)$ [1, 2]. He used Boltzmann's formula $S = k \log W$ and counted the ways of distributing discrete energy elements $h\nu$ over oscillators. A quick check: $\int x^3/(e^x - 1)\,dx$ from 0 to ∞ is $6.4939394 = \pi^4/15$, and the Stefan–Boltzmann constant $\sigma = 5.670374 \times 10^{-8}$ W m^{-2} K^{-4} and Wien's constant $b = 2.897772 \times 10^{-3}$ m·K computed from the law both equal the CODATA values. The derivation rests on Boltzmann's $S = k \log W$ (see the chapter on statistical mechanics).

In 1905 Einstein proposed light quanta: light itself carries energy in units proportional to its frequency [3, 2]. This explains the threshold of the photoelectric effect. Planck's work of 1900 influenced the idea but was not used in a proof.

In 1913 Bohr found the stationary states of hydrogen [4, 3]. Applying Planck's quantum of action to Rutherford's nuclear atom gives a discrete set of stable energy states. A quick check: with the Rydberg constant $R\infty = 10\ 973\ 731.568160$ m^{-1} (infinite nuclear mass), the ground state is $E_1 = -13.6057$ eV and the Hα line ($3 \to 2$) is at 656.11 nm; with the reduced mass of the electron it is at 656.47 nm. The model rests on Planck (1900) and Einstein (1905).

In 1925 Heisenberg turned the Fourier amplitudes of electron orbits into arrays with a non-commuting product; his paper was received on 29 July 1925 [3, 5]. Born recognised the product as matrix multiplication, and with Jordan derived $XP - PX = i\hbar$ in a paper received on 27 September 1925 (note 1). A quick check: a truncated oscillator with $N = 6$ levels gives the diagonal of $[X, P]/i$ as $[1, 1, 1, 1, 1, -5]$, with trace 0. Since the trace of a commutator of finite matrices is always zero, $[X, P] = i \cdot 1$ needs infinite matrices. The step grew out of the failure of the Bohr–Sommerfeld model, guided by the correspondence principle; this was a heuristic, not a use of a proved result.

In the summer of 1925 Dirac read the proofs of Heisenberg's paper [6, 5]. He saw the analogy between commutators and the Poisson brackets of classical mechanics, and

replaced {u, v} by (uv − vu)/iħ. The step rests on Heisenberg's paper of 1925.

In January 1926 Pauli derived the spectrum of hydrogen from matrix mechanics, before wave mechanics existed [5, 7]. He solved it algebraically, using the conserved Runge–Lenz vector and its commutation relations with the angular momentum ℓ. The derivation rests on matrix mechanics.

Wave mechanics, probability and uncertainty

In 1926 Schrödinger set up his wave equation as an eigenvalue problem for a differential operator, starting from de Broglie's matter waves [8, 5]. He showed that his operator functions of q and p obey the rules of Born and Heisenberg, so the two mechanics are equivalent. A quick check: H = −½ d²/dx² + x²/2 on [−10, 10], with 1501 grid points, a three-point Laplacian and zero values at the ends, gives eigenvalues 0.5, 1.5, 2.4999 and 3.4999, that is n + ½. The equivalence theorem rests on matrix mechanics of 1925.

In 1926 Born gave the statistical interpretation: $|\psi|^2$ is a probability density [3, 9]. He applied it to Schrödinger's solutions for scattering, and received the Nobel Prize for it in 1954. The interpretation rests on Schrödinger's wave mechanics.

Heisenberg's uncertainty relation of 1927 was only semi-quantitative (note 2). In 1927 Kennard proved σ(x)σ(p) ≥ ħ/2, and in 1929 Robertson proved the general form σ(A)σ(B) ≥ ½|⟨[A, B]⟩| from the commutator [10, 11, 12]. The standard derivation applies the Cauchy–Schwarz inequality to (A − ⟨A⟩)ψ and (B − ⟨B⟩)ψ. Schrödinger added the covariance term in 1930. A quick check: for ψ = e^(−x²/2) on [−10, 10] with 1001 points, ħ = 1 and the momentum computed by central differences, σ(x)σ(p) = 0.499950 ≈ ½; with 2001 points it is 0.499988. The bound rests on Born and Jordan's [Q, P] = iħ and makes Heisenberg's relation of 1927 precise. Weyl found it independently of Kennard, in his book of 1928, crediting Pauli.

In 1928 Dirac found a relativistic equation for the electron of first order in time [13, 6]. Its coefficients are matrices satisfying {γ^μ, γ^ν} = 2η^(μν). Dirac read some of its solutions as a positively charged twin of the electron, the positron, which was later found in experiment. A quick check: the γ matrices in the Dirac representation satisfy all 16 of these Clifford relations, with η = diag(1, −1, −1, −1). The equation rests on Schrödinger's wave mechanics and on special relativity (see the chapter on relativity).

Operators and Hilbert space

Between 1927 and 1932 von Neumann laid the Hilbert-space foundations of quantum mechanics [16, 17, 18, 19]. His three papers of 1927 gave the formalism and a proof, without delta functions, that matrix and wave mechanics are equivalent; his book followed in 1932. His "proof" that no hidden variables exist assumes that the values assigned to non-commuting observables combine linearly. The work rests on matrix and wave mechanics.

In 1930 Stone stated, and in 1931 von Neumann proved in full, that every irreducible pair of unitary groups satisfying the Weyl relations is equivalent to the Schrödinger representation [14, 15]. Stone's theorem turns the self-adjoint operators Q and P into unitary groups. Wielandt showed that Q and P cannot both be bounded. The theorem

rests on Heisenberg's commutation relation QP − PQ = iI, which posed the uniqueness problem, as Rédei explains [15].

In 1951 Kato proved that atomic Hamiltonians with Coulomb potentials are self-adjoint, because the potential is relatively bounded by the Laplacian [20, 21, 22, 23]. The paper appeared in the *Transactions of the AMS*; refereeing took over three years, and von Neumann was a referee. Von Neumann's framework had assumed self-adjointness; according to Simon, von Neumann thought the problem "impossibly hard" (note 5) [22]. Kato's proof uses Rellich's perturbation theorem of 1939 for relatively bounded operators, now called the Kato–Rellich theorem.

In 1967 Dyson and Lenard proved the stability of matter: the energy of N electrons and nuclei satisfies E ≥ −C·N [27, 28]. Their estimates were long, with C of order $10^{14}z^2$. In 1975 Lieb and Thirring gave a new, simpler proof, with a kinetic-energy inequality for fermions together with Thomas–Fermi theory and Teller's theorem that Thomas–Fermi atoms do not bind. Bosons are not stable in this sense: their energy goes as −N^(7/5). A quick check: N^(7/5)/N at N = 10^6 is 251.2, and it grows without bound. Both proofs rest on the Pauli principle; Fisher and Ruelle had formulated the problem in 1966. Lieb and Thirring's proof was new and did not build on Dyson and Lenard's.

Measurement and contextuality

In 1957 Gleason proved that in dimension 3 or more every probability measure on the projections of a Hilbert space has the form Tr(ρP) for a density operator ρ [24, 17]. The proof works with frame functions on the sphere in $\mathbb{R}^3$ and then extends to higher dimensions. It rests on Born's rule of 1926 and on von Neumann's formalism; Mackey had posed the question.

In 1967 Kochen and Specker proved that no noncontextual assignment of values is possible [24, 17]. They gave 117 rays in $\mathbb{R}^3$ that admit no {0, 1}-colouring with exactly one 1 in each orthogonal basis. Later sets used 33 rays (Peres) and 18 rays in $\mathbb{R}^4$ (Cabello and co-workers, 1996). A quick check: for the 18-vector set, 9 bases each need exactly one 1, so the total is 9, which is odd; but each vector lies in 2 bases, so the total must be even. Gleason's theorem was a motivation only. Gleason plus a compactness argument implies that a finite set exists, but Kochen and Specker construct one explicitly (note 3).

Nonlocality and complexity

In 1964 Bell proved that local hidden-variable models obey a bound on correlations that the singlet state violates [25, 26, 17]. In 1966 he also showed that von Neumann's linearity assumption fails for incompatible observables such as σx and σy. The theorem answers the problem posed by Einstein, Podolsky and Rosen in 1935, and it is a counterexample to von Neumann's no-go argument.

In 1969 Clauser, Horne, Shimony and Holt rewrote Bell's argument as a testable inequality with two settings on each side, |S| ≤ 2 [25, 26]. A quick check: the maximum of |S| over all 16 deterministic local strategies is 2. The inequality rests on Bell's theorem of 1964.

In 1980 Tsirelson proved that quantum correlations obey |S| ≤ 2√2, by bounding the

operator norm of the CHSH operator [25, 26]. A quick check: for the optimal settings the largest eigenvalue of that operator is 2.8284271 = 2√2. The bound answers the question posed by the CHSH inequality.

In 2015 the groups of Hensen, Giustina and Shalm carried out loophole-free Bell tests [25, 29]. They violated the CHSH inequality with both the locality and the detection loopholes closed. The Nobel Prize of 2022 went to Aspect, Clauser and Zeilinger. These are experiments, not theorems, and mark the status of the CHSH line.

In 2020 Ji, Natarajan, Vidick, Wright and Yuen proved MIP* = RE [30, 31, 32]. The halting problem reduces to computing the entangled value of nonlocal games, through a compression theorem for such games, so named by Lin in 2025. As a consequence, Tsirelson's problem has a negative answer and Connes' embedding conjecture is false, through their known equivalence. The proof was posted on arXiv on 13 January 2020 (version 3 on 4 November 2022), with a research highlight in *Communications of the ACM* 64(11), 131–138, November 2021 (note 4).

In 2025 Lin proved MIPco = coRE for the commuting-operator model, by adapting the compression theorem of MIP* = RE [33, 34, 35]. The paper was posted on arXiv on 8 October 2025 and appeared at STOC 2026, where it won the Best Student Paper award. Lin defended a PhD thesis with the same title at the University of Amsterdam on 8 September 2026 [39]. The proof builds on MIP* = RE.

The open problem

In 2000 the Clay Institute posed the Yang–Mills existence and mass gap problem as one of its Millennium Problems [36, 37, 38]. No one has proved that quantum Yang–Mills theory on $\mathbb{R}^4$ exists and has a positive mass gap. The problem was open on 28 October 2026, and no link in this chain leads to it.

Notes

1. Wikipedia dates Born's recognition of matrices to July 1925, while MacTutor speaks of a collaboration "which began in 1926"; the receipt dates of the papers settle it for 1925 [5]. $XP - PX = i\hbar$ and $pq - qp = h/2\pi i$ are the same relation with opposite sign conventions.
2. MacTutor says Heisenberg "discovered" the uncertainty bound, while the Stanford Encyclopedia calls the claim that he proved it "certainly wrong"; the proof goes to Kennard (1927) and Robertson (1929) [10].
3. Specker hinted at the Kochen–Specker result in 1960 and Bell published a version in 1966, but the name follows the finite proof of 1967 [17].
4. No refereed full journal version of MIP* = RE was found as of 29 October 2026: arXiv version 3 (November 2022) has no journal reference, and only the 2021 CACM highlight was found [30, 32].
5. That Kato filled a gap that von Neumann's framework left open rests on Simon alone [22].
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39. University of Amsterdam, PhD defence of Randy Lin, MIPco = coRE (8 September 2026)

Statistical mechanics

Statistical mechanics starts with Daniel Bernoulli's kinetic picture of a gas in 1738. It runs through Maxwell (1860) and Boltzmann (1872, 1877) and then splits into two lines. The kinetic line goes from the H-theorem and the objections of Loschmidt and Zermelo to Lanford's short-time theorem of 1975 and the long-time derivation by Deng, Hani and Ma (2024–25), which addresses Hilbert's sixth problem for hard spheres. The lattice line runs from Lenz and Ising and from Peierls through Onsager, Yang and Lee–Yang to Smirnov, the results of Aizenman and Duminil-Copin, and the random-field theorem of 2024.

Three results stand beside the chain rather than in it. Gibbs' ensembles are a parallel framework, not a step from the H-theorem. Einstein and Smoluchowski found the theory of Brownian motion independently, and so did Mermin–Wagner and Berezinskii for the absence of order in two dimensions. The chapter follows 29 links, from Bernoulli in 1738 to the exposition by Bodineau, Gallagher, Saint-Raymond and Simonella in 2026.

The kinetic theory of gases

In 1738, in chapter 10 of his *Hydrodynamica*, Daniel Bernoulli explained the pressure of a gas by the impacts of its molecules [1, 2]. The momentum carried to the wall by many molecules gives a pressure proportional to $N\, m\langle v^2\rangle/V$. A quick check: a simulated Maxwell gas of 10^6 nitrogen molecules ($m = 28.0134$ u) at 300 K gives $N\, m\langle v^2\rangle/3$ divided by NkT equal to 1.00070. Nothing earlier in the chain is needed.

In 1857 Clausius described heat as "the motion which we call heat", after Krönig's paper of 1856 [1, 3, 4]. In 1858 he introduced the mean free path, from the probability of a collision per unit length of path, in *Annalen der Physik* 105, 239–258 (note 3). He did so to answer an objection: fast molecules should mix gases far more quickly than is seen.

In 1860 Maxwell found the velocity distribution of a gas, which Gyenis calls "the first-ever statistical law in physics" [1, 5, 3, 6, 46]. He assumed that the distribution is isotropic and that the three velocity components are independent; this gives a Gaussian for each component. The argument from stationarity under collisions followed in 1867. A quick check: for nitrogen at 300 K, numerical integration gives $\int f = 1.0000$ and a mean speed $\langle s\rangle = 476.17$ m/s, equal to $\sqrt{(8kT/\pi m)}$. Maxwell used Clausius's mean free path of 1858 to compute the viscosity of a gas [5, 6]; Flamm notes that he derived his own form of the free path [3].

Boltzmann's H-theorem and its objections

In 1872 Boltzmann derived his transport equation and the H-theorem, $dH/dt \le 0$ [7, 1]. The Stosszahlansatz, the assumption of molecular chaos, closes the equation for the distribution of a single particle. A quick check with a Kac ring of 400 sites and 40 marked edges, with all balls white at the start: the white fraction is 0.56 at $t = 50$

and 0.535 at t = 100, and returns to 1.0 at t = 400, when each ball has passed all 40 marks. The theorem rests on Maxwell's stationarity argument, which Boltzmann had generalised in 1868 (note 4).

In 1876 Loschmidt raised the reversibility objection [7, 8]. If all velocities are reversed, H increases, so dH/dt ≤ 0 cannot follow from reversible dynamics alone. The objection is a counterexample aimed at Boltzmann's theorem of 1872.

In 1877 Boltzmann identified entropy with the logarithm of the number of complexions of a macrostate, that is, with the logarithm of a volume of phase space [7, 3, 9]. He counted the complexions directly. He never wrote "S = k log W" in that form; Planck did (note 2). Sharp and Matschinsky stress that his Ω is not the logarithm of a probability. A quick check: for N = 1000 and n = 300, ln C(1000, 300) = 607.27, against N·H(0.3) = 610.86, which agree to leading order by Stirling's formula. The paper followed his first reply of 1877 to Loschmidt and was motivated by the objection; whether it really answers it is disputed (note 1).

In 1896 Zermelo raised the recurrence objection [8, 10]. By Poincaré's recurrence theorem, H must rise again after a long enough time. The objection rests on that theorem (see the chapter on dynamical systems, MSC 37).

In 1902 Gibbs set up ensembles: probability measures on phase space for general systems [8, 11, 12]. His argument that entropy increases works by coarse-graining and is separate from Boltzmann's. The framework is parallel to the H-theorem and does not rest on it.

Brownian motion

In 1905 Einstein balanced osmotic pressure against Stokes drag and obtained diffusion with a mean square displacement proportional to t, and D = kT/(6πηr) [13, 14, 15, 16]. In 1906 Smoluchowski found the same result independently, with an extra factor of 64/27. According to Wikipedia, Einstein's papers prompted Smoluchowski to publish (note 7). A quick check: for a particle of radius 0.5 µm in water at 20 °C (η = 1.002 mPa·s), D = 4.286 × 10^{-13} m^2/s, and the root-mean-square displacement in one dimension after 1 s is 0.93 µm. A ±1 random walk of 10 000 walkers has mean square displacement 99.7, 402.0 and 998.6 at t = 100, 400 and 1000, proportional to t.

In 1908–09 Perrin measured the displacements of Brownian particles [13, 17]. They confirmed Einstein's theory and gave a value for Avogadro's number; Chaudesaigues also made such measurements in 1908. Perrin received the Nobel Prize in 1926. The work is an experimental test of Einstein's theory of 1905 (note 5).

In 1923 Wiener constructed Brownian motion rigorously, as a measure on continuous paths (see the chapter on probability) [13, 20, 16]. The construction rests on Einstein's paper of 1905, which posed the problem; Wiener's paper of 1921 starts from Einstein's Gaussian law.

In 1931 Onsager proved the reciprocal relations L_ij = L_ji [21, 22, 23]. They follow from the microscopic time reversibility of fluctuations near equilibrium. He received the Nobel Prize in Chemistry for this in 1968. The paper uses Boltzmann's relation S = k log W for the probabilities of fluctuations, which it invokes in its section 5.

The Ising model, from one dimension to Onsager

In 1920 Lenz posed the Ising model, and in 1925 Ising showed that the one-dimensional chain has no phase transition [18, 19] (note 6). Its free energy, computed with a transfer matrix, is analytic for every T ≠ 0. A quick check: the transfer-matrix magnetisation m(h) = sinh βh/√(sinh² βh + e^(−4β)), with J = 1, tends to 0 as the field h → 0 at β = 0.5, 1 and 2; at h = 10^{-12} it is at most 1.1×10^{-10}. Nothing earlier in the chain is needed.

In 1936 Peierls proved that the two-dimensional model has spontaneous magnetisation at low temperature [18, 19]. He counted contours: a contour of length ℓ has probability at most e^(−2βℓ), and there are at most 3^ℓ such contours from a given start. The proof answers the question posed by Ising's one-dimensional result; Ising had conjectured that there is no transition in any dimension.

In 1964 Griffiths corrected the estimates in Peierls' contour argument [27, 28] (note 8). A quick check: 1936 to 1964 is 28 years. The work repairs Peierls' proof of 1936.

In 1941 Kramers and Wannier found a duality that maps the high-temperature and low-temperature expansions onto each other [24, 25]. Assuming that the singularity is unique, it gives β_c = ½ ln(1 + √2); this is not a proof of the transition point. A quick check: β_c = 0.4406868, sinh(2β_c) = 1.0000000, and tanh β_c = e^(−2β_c) = 0.4142136. Nothing earlier in the chain is needed.

In 1944 Onsager computed the exact free energy of the two-dimensional square-lattice Ising model in zero field, by diagonalising the transfer matrix [24, 25]. He had announced it in 1942; Kaufman gave a simpler derivation in 1949. A quick check with J = 1: Onsager's integral gives −βf = 0.9296954 at β_c. At β = 0.3 the largest transfer-matrix eigenvalue of periodic strips of width 3 to 8 gives values from 0.80256 down to 0.79066, converging to Onsager's 0.790559. Kramers and Wannier influenced the work, since both give the same β_c, but the proof is not said to use their duality.

Magnetisation, zeros and continuous symmetry

In 1952 Yang proved the formula for the spontaneous magnetisation, M = (1 − $\sinh^{-4}$ 2β)^(1/8), by perturbing the transfer matrix and evaluating Toeplitz determinants [18, 24]. Montroll, Potts and Ward simplified the proof in 1963. A quick check: M = 0.91132, 0.97361 and 0.99928 at β = 0.5, 0.6 and 1.0. Onsager had announced the formula in 1948–49 without proof, so Yang proved a conjecture.

In 1952 Lee and Yang proved the circle theorem: the zeros of the partition function of a ferromagnetic Ising model, as a polynomial in z = e^(−2βh), lie on the unit circle |z| = 1 [26, 24]. Neither source describes the method of proof. A quick check: on a 3 × 3 periodic lattice (J = 1) at β = 0.4, all 9 zeros of the polynomial of degree 9 in z have |z| = 1 to within 10^{-15}. No source says the proof uses Yang's paper of the same year; the two belong to one programme but stand independently.

In 1966 Mermin and Wagner, and in 1967 Hohenberg, proved that a continuous symmetry cannot be broken spontaneously at nonzero temperature in dimension d ≤ 2 when the interactions are of short range [29, 30]. In two dimensions the correction from spin waves diverges logarithmically. According to Wikipedia, Berezinskii found the result independently in 1970 (note 7). It stands without an earlier link in this

chain.

The kinetic line: from Lanford to Hilbert's sixth problem

In 1975 Lanford proved that the Boltzmann equation follows from the dynamics of hard spheres in the Boltzmann–Grad limit, for a short time [31, 32, 33, 34]. The proof works within the BBGKY hierarchy and uses a series expansion, as Valente describes. A quick check: 1975 − 1872 = 103 years after Boltzmann's equation. The theorem answers the problem that Boltzmann's work of 1872 posed.

In 2024 Deng, Hani and Ma derived the Boltzmann equation from hard spheres for as long as the Boltzmann solution exists [33, 41, 42, 34]. The proof uses layering in time, a cumulant ansatz and combinatorial control. It was posted on arXiv on 14 August 2024 (version 3 on 18 July 2025) and accepted by the *Annals of Mathematics* on 31 October 2025. A quick check: 1975 to 2024 is 49 years. It extends Lanford's theorem to any time at which the Boltzmann solution is regular, as Bodineau and co-authors also state.

In 2025 Deng, Hani and Ma took the kinetic route to Hilbert's sixth problem [43, 42, 44]. They derived the compressible Euler and incompressible Navier–Stokes–Fourier equations from hard spheres on two- and three-dimensional tori. The paper was posted on arXiv on 3 March 2025 and had no journal reference as of 28 October 2026. A quick check: 1900 to 2025 is 125 years since Hilbert posed the problem. The authors call it "an extension of our previous work", so it rests on their result of 2024.

In 2026 Bodineau, Gallagher, Saint-Raymond and Simonella gave an exposition of the Deng–Hani–Ma proof [34, 45]. It explains what removes Lanford's restriction to short times, and gives no new proof. It is Séminaire Bourbaki exposé 1247 (January 2026, in *Astérisque*), posted on arXiv on 4 February 2026, with a second version on 14 October 2026. It rests on Deng, Hani and Ma (2024).

Critical behaviour and triviality

In 1982 Aizenman and Fröhlich proved, independently, that the scaling limits of the Ising and φ^4 models are trivial in dimension $d \geq 5$ [24, 19]. Aizenman used the random-current representation, Fröhlich the random-walk representation of Brydges, Fröhlich and Spencer. Nothing earlier in this chain is used.

In 2007 Smirnov proved that the critical two-dimensional Ising model is conformally invariant, using discrete holomorphic observables [35, 36, 24, 37]. With Chelkak he proved universality. The paper was posted on arXiv in 2007 and published in the *Annals of Mathematics* 172 (2010), 1435–1467. It rests on Onsager's solved model, which posed the problem of the scaling limit.

In 2015 Aizenman, Duminil-Copin and Sidoravicius proved that the three-dimensional Ising transition is continuous: the spontaneous magnetisation $m^*(\beta_c)$ is zero [38, 24, 39]. The proof uses random currents and the switching lemma, and appeared in *Communications in Mathematical Physics* 334(2), 719–742. It stands without an earlier link in this chain.

In 2021 Aizenman and Duminil-Copin proved marginal triviality in four dimensions: the critical scaling limits are Gaussian [24, 19]. The proof uses intersection estimates for random currents and a multi-scale analysis, and appeared in the *Annals of*

Mathematics. It completes the work of Aizenman and Fröhlich of 1982, which had left d = 4 open for “forty years”.

In 2024 Ding, Liu and Xia proved long-range order for the three-dimensional random-field Ising model at weak disorder, for every temperature below T_c [40, 19]. The paper was posted on arXiv in 2022 and appeared in *Inventiones* in 2024. It extends the results of Imbrie (1985) and of Bricmont and Kupiainen (1988), which held only at very low temperature.

Notes

1. Boltzmann’s paper on entropy and probability is dated 1877 (*Wiener Berichte* 76, 373–435) by Sharp and Matschinsky and by Flamm; Wikipedia’s date of 1871 refers to his earlier work [9, 3]. The Stanford Encyclopedia says the paper followed his first reply to Loschmidt but “hardly ever touches its self-professed main goal”, and Brown and Myrvold call the probabilistic turn “largely beside the Loschmidt point” [7, 10]. Whether it answers Loschmidt remains disputed.
2. The Stanford Encyclopedia and Sharp and Matschinsky say that Boltzmann never wrote S = k log W; Planck wrote it [7, 9].
3. Clausius’ mean free path dates from 1858 (*Annalen der Physik* 105, 239–258; Flamm), not from the 1857 paper as Wikipedia has it [3, 1].
4. Boltzmann’s generalisation of Maxwell’s argument (1868–71) and his transport equation of 1872 are different results, so the differing dates are not a conflict.
5. Perrin’s 1908 paper shows “no sign of Einstein’s work” according to arXiv 2006.05399, but the Nobel citation says his work confirmed Einstein’s theory [16, 17].
6. Lenz is the originator of the model, according to arXiv 2509.00632, Duminil-Copin and Wikipedia [28, 24, 18].
7. Only Wikipedia states that Einstein’s papers prompted Smoluchowski to publish; MacTutor says only that Einstein proposed his theory independently “at roughly the same time” [13, 14]. Only Wikipedia states that Berezinskii found the Mermin–Wagner result independently in 1970, and the original proof technique of Mermin and Wagner was not confirmed [29].
8. Dobrushin’s paper of 1965 (*Theory of Probability and its Applications* 10, 193–213), which proves the transition, was not found described as a repair of Peierls’ argument.
9. Deng, Hani and Ma (2025) has no journal reference yet [43]. Bloch’s theorem, band theory, density functional theory and the work of Gallagher, Saint-Raymond and Texier (2013) are not treated in this chapter.

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Relativity

Relativity starts with Lorentz's transformations of 1892–1904. It runs through Einstein and Poincaré in 1905 and through Minkowski to the field equations of 25 November 1915, and then to the exact solutions of Schwarzschild, Friedmann and Kerr. After 1950 it becomes a theory of partial differential equations: Choquet-Bruhat's local existence theorem, Penrose's singularity theorem, black-hole uniqueness, positive mass, and stability results running from Minkowski space (1993) to slowly rotating Kerr (2022). Kehle and Unger's counterexample to the third law of black-hole mechanics followed in 2022. As of October 2026, the full no-hair theorem without analyticity and the stability of Kerr for large rotation |a| are still open.

Three pairs in this chain are not linked by any arrow. Einstein and Poincaré worked independently. Einstein and Hilbert dispute priority, but neither used the other's proof. Schoen–Yau and Witten gave independent proofs of the same theorem. The chain has 29 links in all, from Lorentz in 1892 to Kehle and Unger in 2022.

Special relativity and spacetime, 1892–1908

Between 1892 and 1904 Hendrik Lorentz found coordinate changes that keep Maxwell's equations in the same form for moving bodies, together with the contraction of lengths. MacTutor calls his books of 1892 and 1895 "the first systematic appearance of the electrodynamic principle of relativity"; the full transformations came in 1904 [2, 1]. A quick check with boosts $\beta_1 = 0.6$ and $\beta_2 = 0.7$ and the form $\eta = \mathrm{diag}(1, -1, -1, -1)$: two boosts compose to $v = (0.6 + 0.7)/(1 + 0.42) = 0.915493$, and $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$ holds to machine precision. Lorentz's work rests on Maxwell's electrodynamics (MSC 78), not on an earlier link in this chain.

In 1905 Albert Einstein derived the Lorentz transformations from two postulates: the principle of relativity and the constancy of the speed of light. He synchronised clocks by light signals; the paper was received on 30 June 1905, and $E = mc^2$ followed later that year [3, 4]. Lorentz's transformations were an influence, but whether Einstein used Lorentz's 1904 paper directly cannot be decided, since his paper cites no one [1, 2].

Henri Poincaré, in a Comptes Rendus note of 5 June 1905 and a Palermo memoir printed in 1906, stated the relativity principle with the group structure of the Lorentz transformations. He read them as rotations in four dimensions with an imaginary time coordinate; the same check as for Lorentz applies [1, 4]. There is no arrow between Poincaré and Einstein. The two found their results independently, and Einstein is known to have read only Poincaré's earlier work of 1900 and 1902 (note 7).

In 1907 Einstein formulated the equivalence principle: free fall is locally the same as inertial motion. From it he obtained gravitational time dilation, in a paper that appeared in 1908 [3, 5].

In 1907–1908 Hermann Minkowski recast the theory as four-dimensional spacetime. The Lorentz group became the orthogonal group O(3,1) of the invariant form η; the

check with boosts β = 0.6 and 0.7 confirms $\Lambda^{T}\eta\Lambda = \eta$. According to MacTutor he "reformulated Einstein's 1905 paper", so this link rests on Einstein 1905 [6]. It also builds on Poincaré's four-dimensional rotations: Minkowski knew Poincaré's second 1905 paper, though the source does not say he used it in a proof [7]. The two dates belong to two works (note 2).

The field equations, 1915–1927

On 25 November 1915 Einstein presented the field equations $G\mu\nu = \kappa T\mu\nu$, generally covariant in the calculus of Ricci and Levi-Civita [3, 5]. Neither Einstein nor Hilbert knew the contracted Bianchi identity in November 1915, according to Pais as quoted by Logunov and colleagues [8]. That identity makes the conservation law $\nabla\cdot T = 0$ automatic. Rowe says Weyl drew this consequence in 1917, while Logunov and colleagues credit Theorem III of Hilbert's 1915 paper (note 9) [9, 8].

The theory passed two classical tests, which can be recomputed. With Mercury's semi-major axis $a = 5.7909050\times10^{10}$ m, eccentricity $e = 0.205630$, period 87.9691 days and $GM_{\odot} = 1.32712440018\times10^{20}$ m³/s², the perihelion advance $6\pi GM/(c^2a(1-e^2))$ comes to 42.98″ per century. With the solar radius $R_{\odot} = 6.957\times10^{8}$ m, the light deflection $4GM/(c^2R_{\odot})$ is 1.751″. Einstein had computed the perihelion on 18 November 1915 from approximate solutions of his 11 November equations (note 3) [54].

The field equations rest on the equivalence principle of 1907, which posed the problem. They use the Ricci–Levi-Civita calculus, which reached Einstein through Marcel Grossmann (MSC 53). They also use Minkowski's spacetime: MacTutor states this, and Scott Walter's history notes that Einstein needed Minkowski's technique on the way to general relativity, "although he did not recognize this for several years" [3, 57].

David Hilbert derived field equations from a variational principle on the metric. He submitted his paper on 20 November 1915, and it was printed in 1916; his proofs dated 6 December 1915 lack the explicit equations [10, 3, 11]. There is no arrow in either direction between Hilbert and Einstein. Priority is disputed, but neither used the other's proof (note 1).

Karl Schwarzschild found the exterior solution for a point mass in 1915 and published it in 1916 [12, 5]. Static spherical symmetry reduces the field equations to ordinary differential equations. A check with sympy on the metric diag$(1-2M/r, -1/(1-2M/r), -r^2, -r^2\sin^2\theta)$ shows that all 16 Ricci components vanish. The Schwarzschild radius of the Sun is $r_s = 2GM_{\odot}/c^2 = 2.953$ km. This link rests on Einstein's field equations.

Jørg Tofte Jebsen in 1921 and George Birkhoff in 1923 proved that every spherically symmetric vacuum solution is static, and hence is Schwarzschild's [13, 14]. No staticity needs to be assumed in advance. Jebsen published two years before Birkhoff, who found the result independently (note 5). The theorem generalises Schwarzschild 1916.

Alexander Friedmann found non-static cosmological solutions in a paper received on 29 June 1922 [15, 5, 16]. A homogeneous, isotropic ansatz reduces the equations to an ordinary differential equation for the curvature radius a(t). A sympy check: for flat dust, a = t^(2/3) with C = 4/9 solves $(\dot a/a)^2 = C/a^3$ with residual 0. The work rests on Einstein's field equations. Einstein objected in a note received on 18 September 1922, and after May 1923 withdrew the objection as "based on an error in my calculations"

[16].

In 1927 Georges Lemaître derived an expanding universe with a velocity–distance relation. He compared a Friedmann-type solution with the redshifts of nebulae, and in 2018 the IAU recommended the name “Hubble–Lemaître law” [17, 18, 19]. There is no arrow from Friedmann: Lemaître found the solutions independently. At the 1927 Solvay conference Einstein told him that Friedmann had had them in 1922 [19].

The PDE theory and black holes, 1952–1975

In 1952 Yvonne Choquet-Bruhat proved local existence and uniqueness for the vacuum Einstein equations [20, 21, 22]. Her key step was wave coordinates, which make the equations hyperbolic. The link rests on Einstein’s equations, which posed the Cauchy problem.

In 1963 Roy Kerr found the rotating vacuum solution from an algebraically special ansatz, as he describes in his own account [23, 24]. A check with $M = 1$ and $a = 0.7$: the horizons $r_\pm = M \pm \sqrt{(M^2 - a^2)}$ give $r_+ = 1.7141$ and $r_- = 0.2859$, and they merge at $a = M$. The solution rests on Einstein’s field equations.

In 1965 Roger Penrose proved that a closed trapped surface, the null energy condition and a non-compact Cauchy surface together imply that some null geodesic is incomplete [25, 26]. The proof combines focusing by the Raychaudhuri equation with the causal structure of spacetime; Penrose received the Nobel Prize in 2020 for it. The Raychaudhuri equation is used in the proof, according to Senovilla [27]. Schwarzschild and spherical collapse were only motivation: Senovilla notes that spherical symmetry had left open whether singularities are generic.

Stephen Hawking then proved a cosmological singularity theorem by reversing Penrose’s trapped surfaces in time on a cosmological scale. He did this first in Physical Review Letters in 1965, then in a series in the Proceedings of the Royal Society in 1966–67, under the strong energy condition [26, 28, 29, 30]. The series appeared in Proc. R. Soc. A 294 and 295 (1966) and A 300 (1967). With Penrose he published the general theorem in A 314 (1970). The link generalises the method of Penrose 1965.

Werner Israel in 1967 and 1968 and Brandon Carter in 1971 proved black-hole uniqueness [31, 32]. Static vacuum black holes are Schwarzschild (1967); the static electrovacuum case followed in 1968 and the axisymmetric case in 1971. The proofs use differential identities and Stokes’ theorem. The full theorem assumes non-degenerate horizons and real analyticity. Schwarzschild and Kerr are the solutions the theorem singles out, so they are its target rather than a step in its proof; Kerr’s uniqueness needed Robinson’s identity, according to Living Reviews [32].

In 1969 Choquet-Bruhat and Robert Geroch proved that a maximal globally hyperbolic development exists and is unique up to isometry [20, 33, 22]. The argument is of Zorn type, over the family of developments. It uses Choquet-Bruhat’s local uniqueness of 1952 in its second step, seventeen years later (1952 → 1969 is 17 years), together with Zorn’s lemma.

In 1973 James Bardeen, Brandon Carter and Stephen Hawking proved laws of black-hole mechanics from Killing horizons, surface gravity and the area theorem [34, 35]. The zeroth, first and second laws were proved, while the third was assumed. The link

rests on Hawking's area theorem of 1971, according to Wald (it is not a separate link here).

Jacob Bekenstein argued in 1972 that black-hole entropy is proportional to area (note 9). In 1974–75 Hawking showed that quantum field theory on the background of a collapse gives thermal emission at a temperature $T \propto 1/M$ [36, 35]. A check with the CODATA values of ħ, k and c and $GM_\odot = 1.32712440018\times10^{20}$ m³/s²: for one solar mass, $T_h = \hbar c^3/(8\pi GMk) = 6.17\times10^{-8}$ K. These results rest on the black-hole mechanics of 1973 and on the area theorem. Hawking's radiation turned $\kappa/2\pi$ into a physical temperature.

Positive mass and the third law, 1979–1986

In 1979 Richard Schoen and Shing-Tung Yau proved the positive mass (ADM) theorem [37]. The proof argues by contradiction with minimal surfaces, in dimensions below 8. In 2017 they extended it to all dimensions without a spin assumption (arXiv 1704.05490; Surveys in Differential Geometry 24(1), 441–480) [38, 39]. The theorem rests on the theory of minimal surfaces (MSC 53), not on an earlier link.

In 1981 Edward Witten gave a spinor proof, two years later (1979 → 1981 is 2 years). Solving a Dirac-type equation turns the ADM energy into a manifestly non-negative integral [37, 40]. There is no arrow to Schoen–Yau: Witten's paper in Comm. Math. Phys. 80 (September 1981) is an independent second proof "by a different method" (note 4).

In 1986 Israel published in Physical Review Letters a "proof" that surface gravity cannot reach 0 in finite time [41, 34]. The argument was later shown to contain a flaw. It rests on the black-hole mechanics of 1973, which had assumed the third law.

Global results, 1991–2022

In 1991 Demetrios Christodoulou found the nonlinear memory effect [42, 43]. A burst of gravitational waves leaves a permanent displacement of test masses, and the effect comes from the nonlinearity of the vacuum Einstein equations. It has no arrow to an earlier link in the chain.

In 1993 Christodoulou and Sergiu Klainerman proved the global nonlinear stability of Minkowski space [42, 44, 45]. Their vector-field method uses approximate Killing and conformal Killing fields, together with energy estimates based on the Bianchi identities. This too stands without an arrow to an earlier link.

In 1994 Christodoulou showed that naked singularities exist in the spherically symmetric collapse of a scalar field. In 1999 he showed that they are unstable [42, 46]. The first result was an explicit construction and the second a genericity argument. There is no arrow from the 1994 construction: the 1999 paper uses his earlier collapse results "in an essential way" and cites the 1994 examples only as known cases.

In 2009 Christodoulou proved that a strong incoming flux of gravitational waves forms a trapped surface, by a short-pulse method [42, 47]. Penrose's theorem then gives incompleteness. Senovilla frames the result as resting on Penrose 1965 for that last step, though the trapped-surface proof itself does not use Penrose (note 10) [27].

In 2021 Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski and Martin Taylor proved the nonlinear stability of the Schwarzschild family in codimension 3 [48, 49]. Their proof builds on their own earlier linear stability of the Kerr family near Schwarzschild. It rests on Schwarzschild 1916, which posed the problem. The paper is a preprint: arXiv, 16 April 2021, version 1 only, with no journal reference found on 29 October 2026.

In 2022 Elena Giorgi, Sergiu Klainerman and Jérémie Szeftel proved the nonlinear stability of slowly rotating Kerr black holes ($|a| \ll m$) [44, 24, 50, 51]. They use GCM foliations and a method that, in their words, “extends the one used in the stability of Minkowski space”. The arXiv paper has 912 pages, and the whole proof about 2,100. It appeared on arXiv on 30 May 2022 and was published in Pure and Applied Mathematics Quarterly 20(7) (2024), pp. 2865–3849. It came 29 years after Christodoulou–Klainerman (1993 → 2022 is 29 years). The link rests on Kerr 1963, which posed the problem, and on Christodoulou–Klainerman’s method, as the authors’ abstract states (note 10).

In 2022 Christoph Kehle and Ryan Unger showed that extremal black holes form in finite time from regular data, which disproves the “third law” [52, 34, 53]. They glue, along characteristics, a Minkowski light cone to an extremal Reissner–Nordström horizon, and choose the pulses with the Borsuk–Ulam theorem. The paper appeared on arXiv on 28 November 2022 (v2 on 15 February 2024), was accepted on 30 September 2023, and was published online in the J. Eur. Math. Soc. on 27 January 2025. It is a counterexample to Israel 1986, whose argument has a flaw, and it uses Borsuk–Ulam (MSC 55). It came 49 years after the third law was assumed (1973 → 2022 is 49 years).

Notes

1. MacTutor’s Einstein page says Hilbert submitted the correct field equations a week before Einstein [3]. Its Hilbert page, citing Corry, Renn and Stachel (1997), says his proofs of 6 December lack them and that the printed version concedes Einstein’s priority [10]. The proofs, as reported, favour Einstein.
2. Minkowski’s dates belong to two works, not a conflict: the 1907–08 paper on moving bodies and the 1908 Cologne lecture “Raum und Zeit” [7].
3. Mercury’s perihelion is dated 18 November 1915, from approximate solutions of the 11 November equations, not October. Weinstein writes “On November 18, 1915, Einstein presented” and that he “tried to find approximate solutions to his November 11, 1915 field equations” [54].
4. Witten’s paper is in Comm. Math. Phys. 80, September 1981 [40].
5. Jebsen (1921) came before Birkhoff (1923), so the name “Birkhoff’s theorem” is an attribution conflict [13, 14]. The proof mechanism (a Killing field) was not in either source and is not described here.
6. Wikipedia says Bekenstein coined “no hair”. Bekenstein’s own 1996 review says “Wheeler enunciated the Israel–Carter conjecture, today colloquially known as ‘black holes have no hair’ ”, citing Ruffini and Wheeler 1971; it does not say who first used the phrase [31, 55].
7. Priority for special relativity remains disputed. Whittaker credits Poincaré, while Zahar and Darrigol speak of independent discovery [1, 4].
8. Kruskal found his extension of Schwarzschild in 1956–57, and Wheeler published it under Kruskal’s name in Phys. Rev. 119 (1960), 1743–1745, so both 1957 and

1960 are correct [56].

9. Who first brought the contracted Bianchi identity into the theory is disputed: Rowe says Weyl in 1917, Logunov and colleagues say Hilbert's 1915 Theorem III [9, 8]. Bekenstein's entropy is dated 1972 by Wikipedia but cited as 1973–74 by Wald [36, 35].
10. Some dependencies rest on one source. Christodoulou 2009 as resting on Penrose's theorem is Senovilla's framing [27]. Giorgi–Klainerman–Szeftel's use of the Christodoulou–Klainerman method is stated only in the authors' abstract [44]. The arrow from Kerr to the no-hair theorem marks a target rather than a proof step.
11. Open as of October 2026, and not rechecked for 2025–26: the stability of Kerr for large |a| and the full no-hair theorem without analyticity. The Dafermos–Holzegel–Rodnianski–Taylor paper has no journal reference. Not covered in this chapter: Kruskal–Szekeres coordinates, the Newman–Penrose formalism, Robinson 1975, Hawking's rigidity theorem, Alexakis–Ionescu–Klainerman, the Penrose inequality, Hintz–Vasy and the LIGO detection of 2015.

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Astronomy

The chain starts with Kepler's empirical laws of 1609–1627, fitted to Tycho Brahe's observations of Mars, which Newton derived from the inverse-square law in 1687. From Newton it splits into three lines. The first is celestial mechanics: the secular theory of Laplace and Lagrange, Gauss's orbit of Ceres, the discovery of Neptune, Poincaré, and Laskar's chaos. The second is stellar structure, from the Lane–Emden equation to the Chandrasekhar limit. The third is relativistic astrophysics: Schwarzschild, the Tolman–Oppenheimer–Volkoff equation, Buchdahl, Kerr, Hawking, gravitational-wave templates and GW150914.

In 2026 the chain ends with GW250114 (Physical Review Letters, 10 September 2025). This is an observational test of Hawking's area theorem and of the Kerr ringdown, not a theorem; no rigorous theorem in this field from 2025–26 was found (note 12). No arrow leads into or out of Laplace's nebular hypothesis of 1796. Adams and Le Verrier worked in parallel, with no arrow between them, and Eddington's objection of 1935 is not a counter-proof. The chain has 32 links, from Kepler in 1609 to GW250114 in 2025.

From Kepler to Newton, 1609–1713

In *Astronomia nova* (1609) Johannes Kepler stated his first law, that the orbit of Mars is an ellipse with the Sun at one focus, and his second law of equal areas [1, 2]. The second law was not yet in its modern form there; that came in 1621 (note 1). Both laws were fitted to Tycho Brahe's observations of Mars. They were empirical, not proved, and they rest only on Tycho's data.

In *Harmonices mundi* (1619) Kepler stated the third law, $T^2 \propto a^3$, an empirical regularity across the planets [1, 2]. A check with the JPL J2000 mean semi-major axes (0.38709927, 1.00000261, 5.20288700 and 30.06992276 AU) and sidereal periods (0.2408467, 1.0000174, 11.862615 and 164.79132 years): T^2/a^3 comes to 1.0000 for Mercury, 1.0000 for Earth, 0.9991 for Jupiter and 0.9988 for Neptune.

In 1627 Kepler published the Rudolphine Tables, computed from Tycho's data [1, 3, 4]. This is a computational link, not a theorem. It rests on Tycho's observations and on Kepler's laws: MacTutor says the tables used "Kepler's first two laws", and Cambridge HPS refers to "elliptical planetary orbits".

Isaac Newton proved that equal areas are equivalent to a central force, and that an ellipse with the force at a focus implies a force proportional to $1/r^2$ [2, 5, 6]. He did this in *De motu* (1684) and in the *Principia* (1687), by geometric limit arguments, and he added universal gravitation. A numerical check (scipy DOP853, rtol = atol = 10^{-12}; GM = 1, start at r_0 = (1, 0) with v_0 = (0, 1.2), so a = 1.785714 and T = 2πa^(3/2) = 14.9933): the areal velocity stays constant to about 10^{-10}, and the orbit returns to its starting point to 5.9×10^{-11} after one period. The link rests on Kepler's first and second laws. Hooke's claim of 1686 is a priority claim, not a proof step. Halley's question of 1684 about the orbit under an inverse-square force prompted *De motu* (note 5) [7, 8, 9].

The converse, that an inverse-square force gives conic orbits, came later [5, 6, 10, 11]. The first edition of the *Principia* asserted it without proof. The second edition of 1713 sketches a proof in Corollary 1 to Proposition 13 of Book I, according to MacTutor, added after criticism. Jakob Hermann and Johann Bernoulli gave proofs by the calculus in 1710. This link repairs a gap in Newton 1687: the first edition "asserted, without proof" that conics are the only solutions (Nauenberg), and "no justification is offered" (Grugnetti).

Secular theory and the planets, 1773–1846

In 1773 Pierre-Simon Laplace showed by averaged perturbation that the semi-major axes have no secular term at first order [12, 13, 14]. The paper was read in 1773 and published in 1776. It rests on Newton 1687.

Between 1774 and 1784 Joseph-Louis Lagrange, together with Laplace, derived linear secular equations with constant coefficients for the nodes, inclinations and eccentricities [14, 13]. Real eigenfrequencies give quasi-periodic solutions. A check with the Laplace–Lagrange eccentricity matrix in the form of Murray and Dermott, for Jupiter and Saturn alone (masses 1/1047.3486 and 1/3497.9018 $M_\odot$, a = 5.202545 and 9.554841 AU, periods 11.862615 and 29.447498 years, and Laplace coefficients $b^{(1)}$ and $b^{(2)}$ of order 3/2 by quadrature): the eigenfrequencies are 3.472 and 22.011″ per year, both real, with periods of 373,284 and 58,881 years. These have not been compared with a published two-planet value. The work builds on Laplace 1773; how the credit divides between the two men is unclear (note 4).

In 1784–1786 Laplace showed that eccentricities and inclinations "always remain small", which is stability at first order [12, 13, 14, 15]. He also explained the great inequality of Jupiter and Saturn by their near-commensurability. This rests on Lagrange's secular equations and on Newton 1687. Euler's prize theory of 1748 had posed the problem of the great inequality. According to Laskar, Lagrange had attacked it in a memoir of 1766, which found Jupiter accelerating and Saturn slowing, as observed, but with calculations that were still incorrect [14]. Laskar dates Laplace's explanation to 1785, Musielak to 1786 (note 9).

In 1796 Laplace put forward the nebular hypothesis in *Exposition du système du monde* [12, 16]. It was qualitative and had no proof. No arrow leads into it or out of it.

In 1801 Carl Friedrich Gauss computed the orbit of Ceres from only 9° of arc [17, 18, 19]. He fitted a Keplerian conic through three observations. Franz Xaver von Zach recovered Ceres in December 1801: on 7 December according to MacTutor, on 31 December (with Olbers) according to Wikipedia (note 10). Gauss used least squares without disclosing it, and published the method in *Theoria motus* in 1809. A check: 7 December is 340 days after Piazzi's sighting of 1 January 1801 (31 December is 364 days), and 9° is 0.025 of a circle. The link rests on the two-body conic of Newton 1687.

In 1840–41 Urbain Le Verrier showed that higher-order secular expansions for the inner planets have denominators close to zero, the problem of small divisors [14, 20]. This came 52 years before Poincaré's work of 1892. It extends the Lagrange–Laplace secular theory beyond the linear approximation (Laskar 2012; arXiv 1710.03688).

In 1845–46 Le Verrier and John Couch Adams predicted an unseen planet by inverting the residuals of Uranus under Newtonian gravity [21, 22]. Johann Galle found Neptune

on 23 September 1846, less than 1° from Le Verrier's position. That was 23 days after Le Verrier's memoir of 31 August. For comparison, Bode's law gives $0.4 + 0.3 \times 2^7 = 38.8$ AU, against the actual 30.07 AU (JPL J2000 mean a). The prediction rests on Newton's gravitation, which both men applied to the irregularities of Uranus. There is no arrow between Adams and Le Verrier: their work was parallel.

Chaos and the stability of the solar system, 1889–2009

In 1889–90 Henri Poincaré proved that the three-body problem has no further analytic integrals [25, 14, 26, 27]. Homoclinic points make the motion chaotic, and the astronomers' series diverge. Lars Phragmén found an error in the prize version, and the corrected memoir of 1890 revealed the homoclinic tangles. This came 104 years after Laplace's stability result of 1786. Poincaré's theorem is a counter-result to the Lagrange–Laplace stability proofs. It extends Bruns's theorem of 1887, which excluded further algebraic integrals, to analytic ones. Lagrange's prize work of 1772 on the three-body problem is background, but no source links it to Poincaré.

Andrey Kolmogorov in 1954 and Vladimir Arnold in 1963 proved the KAM theorem: quasi-periodic tori persist under small perturbations, on a set of positive measure [14, 45, 46]. This partly resolves the small-divisor problem. Laskar notes that it does not apply directly to the real planetary problem. The theorem builds on the small-divisor problem of perturbation theory. Laskar places Kolmogorov's result "in line with the work of Poincaré", but he does not tie KAM explicitly to Le Verrier's small divisors (note 13).

In 1989 Jacques Laskar showed that the solar system is chaotic, with a Lyapunov time of about 5 million years [45, 14]. In 2009 Laskar and Mickaël Gastineau found that in about 1% of 2,501 neighbouring futures Mercury reaches a large eccentricity within 5 billion years. Both results came from numerical integration of averaged secular equations. A check: with a Lyapunov time of 5 Myr, an error of 15 m grows to 1 AU after $5 \times \ln(1\ \text{AU} / 15\ \text{m}) = 115.1$ Myr, consistent with "about 100 Myr". The work uses the Lagrange–Laplace secular method.

Stars, 1870–1939

In 1870 Jonathan Homer Lane derived the equation $\theta'' + (2/\xi)\theta' + \theta^n = 0$ for polytropic gas spheres, now called the Lane–Emden equation [23, 24]. It follows from hydrostatic equilibrium, Poisson's equation and the relation P = Kρ^(1+1/n) (Lane, American Journal of Science, 1870; Emden's date is not given in either source). A check with scipy solve_ivp (rtol 10^{-12}, started at $\xi = 10^{-6}$ from the series): the first zeros are π for n = 1, 3.6537537 for n = 1.5 and 6.8968486 for n = 3, with $\omega_3 = 2.0182360$. The n = 3 values match Ambrosino 2020 (6.89685 and 2.01824). There is no earlier link in this chain behind it.

In 1902 James Jeans found a critical length and mass for gravitational instability [28, 29]. He used a linearised fluid and Poisson's equation on a static background, the so-called "Jeans swindle". Kiessling showed in 1999 that it is justified as a well-defined limit. An illustrative check with T = 10 K, $n = 10^4$ cm^{-3}, μ = 2.33, ρ = μm_H·n, $\lambda_J = \sqrt{(15kT/(4\pi G\mu m_H \rho))}$ and $M_J = (4/3)\pi\rho(\lambda_J/2)^3$: $\lambda_J = 0.131$ pc and $M_J = 0.67\ M_\odot$. The result rests on linearised fluid mechanics (MSC 76).

In 1924 Arthur Eddington derived the mass–luminosity relation from radiative

diffusion, an ideal-gas interior, the virial theorem and radiation pressure [35, 36, 37]. Its simplest form gives $L \propto M^3$, while the observed exponent is about 3.5. For a star of 10 $M_\odot$ that is 1000 $L_\odot$ with exponent 3 against 3162 $L_\odot$ with exponent 3.5. The link rests on the empirical relation suggested by Halm in 1911 and discussed by Russell in 1913 and Hertzsprung in 1918 (note 7).

In 1926 Ralph Fowler explained white dwarfs as a completely degenerate Fermi–Dirac electron gas, whose pressure balances gravity [38, 39]. This rests on Fermi–Dirac statistics of 1926 (MSC 82).

In 1929–31 Wilhelm Anderson, Edmund Stoner and Subrahmanyan Chandrasekhar showed that relativistic degeneracy gives P ∝ ρ^(4/3) [38, 40, 41, 42]. That is a polytrope of index n = 3, whose mass does not depend on the central density: a limiting mass (note 2). Lev Landau found it independently in February 1931 and published it in 1932 (note 8). A check with $M_{Ch} = \omega_3(\sqrt{(3\pi)}/2)(\hbar c/G)^{(3/2)}/(\mu_e m_H)^2$, $\omega_3 = 2.01824$, $\mu_e = 2$ and CODATA constants: $M_{Ch} = 1.4353\ M_\odot$ with the proton mass for m_H, and 1.4338 $M_\odot$ with the mass of the hydrogen atom; Ambrosino gives 1.43469. The result rests on Fowler 1926, which Anderson and then Stoner made relativistic. It uses the Lane–Emden equation: Bonolis's history says Chandrasekhar worked with "the Lane-Emden equations for polytropes" [64]. Landau's work was independent.

In 1935 Eddington rejected relativistic degeneracy, calling for "a law of Nature to prevent a star from behaving in this absurd way" [38, 40]. This was an objection, not a counter-proof, so it carries no arrow as a proof. Chandrasekhar received the Nobel Prize in 1983, 52 years after 1931.

In 1939 Richard Tolman, Robert Oppenheimer and George Volkoff derived the TOV equation for a static, spherical perfect fluid [43, 44]. They found the maximum mass of a star made of a cold neutron Fermi gas and checked it qualitatively against Tolman's analytic solutions. A check (ideal neutron Fermi gas with the exact Chandrasekhar-form P and ε, neutron mass, CODATA constants; scipy solve_ivp with rtol 10^{-10}, surface at $P = 10^{-12} P_c$, maximum over the central Fermi momentum): $M_{max} = 0.710\ M_\odot$ at R = 9.16 km, reached at $p_F/m_n c = 0.834$ (note 3). The link rests on the field equations. The parallel with the Chandrasekhar limit, a limiting mass of a degenerate Fermi gas, is suggested by the content, but no source consulted states it (note 13).

Relativistic astrophysics, 1916–1999

In 1916 Karl Schwarzschild published the exact vacuum field of a point mass, found in 1915 [30, 31]. Static spherical symmetry reduces the field equations to ordinary differential equations; Johannes Droste found the solution independently. A check with $GM_\odot = 1.32712440018\times10^{20}\ m^3/s^2$: r_s for the Sun is $2GM/c^2 = 2953.25$ m = 2.953 km. The solution rests on Einstein's field equations (see the chapter on relativity).

Also in 1916, Schwarzschild found the exact interior solution for a star of constant density [32, 33, 34]. The central pressure becomes infinite when $\cos\chi_a = 1/3$, so the radius R must exceed 9/8 of the Schwarzschild radius. A check of $P_c/\rho c^2 = (1 - \sqrt{(1-x)})/(3\sqrt{(1-x)} - 1)$ at $x = 2GM/Rc^2 = 0.8$, 0.88, 0.888 and 0.8888 gives 1.618, 16.66, 166.7 and 1666.7 as x approaches 8/9; the limit 9/8 r_s is 3.322 km for the Sun. The solution rests on the field equations and is joined continuously to the exterior solution at the surface.

In 1959 Hans Buchdahl proved that $2GM/(Rc^2) \le 8/9$ for static spherical bodies with

non-increasing density and isotropic pressure; the bound is attained by the constant-density star [33, 47]. In 2007–08 Håkan Andréasson (Journal of Differential Equations) replaced both assumptions by a condition on the pressures and proved a sharp bound. A check: 8/9 ÷ 2 = 4/9 for M/R with G = c = 1. Buchdahl's theorem uses the TOV equations, as Arrechea and colleagues state it [65], and generalises Schwarzschild's 9/8 limit.

In 1963 Roy Kerr found the exact field of a spinning mass as an algebraically special metric, as the title of his paper says [48, 49]. Its Weyl tensor is of Petrov type D. A check with M = 1 and a = 0.7: the horizons are at 1.7141 and 0.2859. The solution rests on the field equations.

In 1974–75 Stephen Hawking showed that a black hole emits thermally, at a temperature set by its surface gravity, and evaporates in finite time [50, 51, 66, 52]. He used quantum field theory on the background of a collapse. A check with CODATA ħ, k, c and G, $GM_\odot = 1.32712440018\times10^{20}$ m³/s², and $t = 5120\pi G^2M^3/(\hbar c^4)$ with t = 13.8 billion years of 365.25 days: the exact T_H for 1 M☉ is 6.17×10^{-8} K, and the mass that evaporates in 13.8 billion years is 1.7×10^{14} g. Hawking's letter in Nature estimated the temperature as about 10^{-6} (M☉/M) K (note 6) [66]. The work rests on Bekenstein 1972 and on the superradiance argument that Zel'dovich and Starobinsky gave Hawking in Kraków in September 1973.

From the 1980s to 2024 Luc Blanchet and others derived the post-Newtonian equations of motion of compact binaries to 4PN, and the flux and phasing to 4.5PN (PRL 131, 2023) [53, 54]. The method merges the multipolar post-Minkowskian expansion with the post-Newtonian one (MPM-PN). It has no arrow to an earlier link in this chain.

In 1998–99 Alessandra Buonanno and Thibault Damour introduced the effective-one-body method (Phys. Rev. D 59, 084006, 1999) [55, 56]. It maps the two-body problem onto a test particle in an effective metric by resumming the post-Newtonian results, on which it rests.

Gravitational waves, 2005–2025

In 2005 Frans Pretorius achieved the first numerical evolution of a binary black hole through plunge, merger and ringdown [57, 49]. The remnant was a Kerr black hole with spin about 0.70, and about 5% of the mass was radiated. The link rests on the Kerr metric, by which the remnant is identified.

In 2015 the LIGO and Virgo collaborations made the first direct detection, GW150914 [58, 56, 55, 59]. The matched-filter signal-to-noise ratio was 24, the significance above 5.1σ, and the masses 36 + 29 → 62 M☉ with final spin 0.67. The SEOBNRv2 and IMRPhenomPv2 templates were used. A check with $A = 8\pi(GM/c^2)^2(1 + \sqrt{(1-\chi^2)})$, χ = 0 initially and 0.67 at the end: 3 M☉ was radiated, and the horizon area grew from 234,215 to 367,030 km², a ratio of 1.567. The detection rests on effective-one-body templates calibrated to numerical relativity, and on the Kerr remnant.

In 2025 the LIGO–Virgo–KAGRA collaboration reported GW250114 (PRL 135, 111403), with signal-to-noise ratio 80 [60, 61, 62, 63]. The remnant area exceeds the initial area with high credibility (99.999% according to IFAE and Cornell). The ringdown matches the Kerr mode $\ell = |m| = 2$ and its first overtone within ±30%. This is an observational test of Hawking's area theorem of 1971 and of the Kerr spectrum, not a theorem or a

proof. A check with the same formula, $\chi = 0$ and masses 33.6 + 32.2 $M_\odot$: the initial area is 237,372 km² for non-spinning holes, which the small measured spins ($\chi \leq 0.26$) would lower slightly; IFAE gives about 240,000. The observation tests, but does not use, Hawking's area theorem and the Kerr ringdown spectrum, and it rests on the method of GW150914.

Notes

1. MacTutor puts Kepler's area law in *Astronomia nova* (1609), while Wikipedia puts its modern area form in the *Epitome* (1621) [1, 2]. These are probably two stages of one law; the point has not been settled from a primary source.
2. Anderson and Stoner (1929) had a first limiting mass, for a star of uniform density, before Chandrasekhar. The name of the limit is therefore an attribution conflict, not settled [38, 41].
3. The APS abstract says Oppenheimer and Volkoff found no static solutions above 3/4 $M_\odot$, while Wikipedia gives about 0.7 $M_\odot$ and the integration above gives 0.710 $M_\odot$ at 9.16 km [44, 43]. The 3/4 is probably a rounded bound, but the full text was not opened.
4. MacTutor credits Laplace with the small eccentricities and inclinations, while Laskar credits Lagrange with the linear secular equations ("Laplace was very impressed by the results of Lagrange") [12, 14]. The work is best read as joint (Laplace–Lagrange).
5. Halley's visit to Newton is dated August 1684. King's College London writes "In August 1684 Edmond Halley visited Isaac Newton", and Cambridge University Library says "summer of 1684", with *De motu* received in November 1684 [8, 9]. MacTutor's Halley page gives "August 1682", which is an error [7].
6. Hawking's 1974 estimates in Nature (about 10^{-6} K and 10^{71} s) differ from the exact values (6.17×10^{-8} K, and 6.6×10^{74} s for photon emission only) [66]. Whether his 1975 paper corrected them was not checked.
7. The mass–luminosity relation was suggested by Halm in 1911 and derived by Eddington in 1924, so MacTutor's word "discovered" refers to the derivation [37, 35].
8. Landau's limit is dated February 1931 in the Physics Today letter and 1932 (publication) in Wikipedia [41, 38].
9. Musielak dates Laplace's explanation of the great inequality to 1786, Laskar to 1785 [15, 14].
10. Ceres's recovery is dated 7 December 1801 by MacTutor and 31 December (Zach and Olbers) by Wikipedia [18, 19].
11. Minor or settled points: the Rudolphine Tables are of 1627; Chandrasekhar worked in 1930 and published in 1931; Laplace's invariance paper was read in 1773 and printed in 1776; Le Verrier's small divisors are of 1840–41 (Laskar); Hawking's area theorem is of 1971 (APS Physics, IFAE). The differing white-dwarf limits (1.4, 1.44, 1.43469, 1.4353) and solar Schwarzschild radii (2.953, 2.954 km) come from the choice of constants. Nauenberg says Johann Bernoulli criticised Hermann's 1710 solution, not Newton's Corollary 1. Blanchet's 4.5PN result concerns flux and phasing (PRL 131, 2023), not the full waveform.
12. Not covered: no rigorous theorem in this field from 2025–26 was found (GW250114 is a test). Kerr stability and extremal black holes are treated in the chapter on relativity. Vlasov–Poisson theory and galactic dynamics, the Saha equation, radiative transfer, Hubble, Zwicky and the step from Copernicus to

Kepler are not included.

13. Some statements rest on one source. Only Laskar describes Lagrange's 1766 work on the great inequality [14]. Laskar does not tie KAM explicitly to Le Verrier's small divisors [14]. No source consulted states the analogy between the TOV maximum mass and the Chandrasekhar limit.

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Geophysics

The chain starts with Eratosthenes' measurement of the Earth by shadows, around 230 BC. No arrow leads from it to Newton's oblate Earth and equilibrium tide of 1687. From Newton the chain splits into five lines. The first is the figure of the Earth (Lapland, Clairaut, Stokes, Love numbers); the second is the tides (Laplace, Kelvin, Doodson); the third is the thermal age of the Earth (Kelvin, Perry, Patterson). The fourth is geomagnetism and dynamo theory (Gauss, Larmor, Cowling and Zeldovich, Herzenberg and Backus, mean-field theory, simulations and sodium experiments), and the fifth is inverse theory (Herglotz, Backus–Gilbert). Ekman's spiral forms a smaller line in fluid dynamics.

In 2026 the latest links are two arXiv preprints on necessary bounds for dynamo action: Kaiser and Tilgner (2025) and Arslan and Grayer (2026). None of the links inside this chain is a direct use of one proof in another. Each is a generalisation, a test, a use of a method or a challenge. Patterson's radiometric age does not use Kelvin's conduction argument at all. The chain has 26 links, from Eratosthenes to Arslan and Grayer.

The figure of the Earth and the tides, c. 230 BC – 1849

Around 230 BC Eratosthenes found the circumference of the Earth to be 50 × 5000 = 250,000 stadia [1, 2]. He assumed parallel rays of sunlight, so that the noon shadow angle of 7.2° at Alexandria equals the arc from Syene to Alexandria, 1/50 of a circle. A check: this is 39,300 km at 157.2 m per stadion, or 41,675 km at 166.7 m, against a true value of about 40,075 km (note 1). The argument rests on Greek geometry (MSC 51); the date rests on one cited book (note 13).

In 1687 Isaac Newton showed in the *Principia* (Book III, Proposition XIX) that a homogeneous, rotating fluid Earth is oblate, with flattening f = 1/230 [3, 4, 5]. He also explained the tide by the tidal force of the Moon and Sun [6, 7]. Papadopoulos credits the column-balancing argument to Clairaut rather than to Newton, so it is not claimed for Newton here [5]. A check with WGS84 values ($\omega = 7.292115\times10^{-5}$ s^{-1}, a = 6378137 m, $g_e = 9.7803253359$ m/s^2): $m = \omega^2 a/g_e = 0.0034677$, and $f = (5/4)m = 1/230.7$. Newton's result rests on his gravitation (see the chapter on astronomy). There is no arrow from Eratosthenes.

In 1736–37 Maupertuis, Clairaut, Celsius and others measured a meridian arc in Lapland [8, 9]. Published in *La Figure de la Terre* (1738), it confirmed an oblate Earth. This was an observation, not a proof: a test of Newton's oblate Earth against Cassini's prolate one.

In 1743 Alexis Clairaut proved, by first-order theory for nested spheroidal layers in equilibrium, that $g(\varphi) = g_e[1 + (5/2\cdot m - f)\sin^2\varphi]$ [3, 8]. A check with m as above, the WGS84 flattening f = 1/298.257223563 and the observed $\beta = (g_p - g_e)/g_e$ with $g_p = 9.8321849378$ m/s^2: $5/2\cdot m - f = 0.0053166$, against the observed 0.0053024, a difference of 0.3%. Clairaut's theorem generalises Newton 1687 and uses the Lapland data. Later, according to Wikipedia, Laplace obtained f = 1/330 from 15 gravity values (note 12) [3].

In 1775–76 Pierre-Simon Laplace gave a dynamic theory of the tides [6, 10, 7]. His tidal equations are the linearised shallow-water equations on a rotating sphere with tidal forcing. A check of the equilibrium tide $H = (3/2)(M_{moon}/M_{earth})(R/d)^3 R$, with $M_{moon} = 7.342\times10^{22}$ kg, $M_{earth} = 5.9722\times10^{24}$ kg, $R = 6371$ km and $d = 384{,}400$ km: the amplitude is 0.535 m. The ratio of solar to lunar tide, $(M_{sun}/M_{moon})(d_{moon}/d_{sun})^3$ with $d_{sun} = 1$ AU, is 0.459. Laplace's theory makes Newton's equilibrium tide dynamic.

In 1849 George Gabriel Stokes showed that Clairaut's result holds for any law of density, provided the outer surface is an equilibrium spheroid [3, 14, 15]. His tool was potential theory: the exterior field is fixed by the bounding surface and the rotation. This generalises Clairaut 1743 "without making any hypothesis whatsoever respecting the distribution of matter in the interior".

Geomagnetism, 1832–1839

In 1832 Carl Friedrich Gauss, working with Wilhelm Weber, made the first absolute measurement of magnetic intensity [11, 12, 13]. In 1839 he analysed the geomagnetic field in spherical harmonics, using Laplace's equation for the magnetic potential fitted to worldwide observations. He showed that there are only two poles. A toy check (numpy, seed 42: 500 points with $\cos\theta$ uniform on $[-1, 1]$, an internal axial dipole $g_1^0 = -29400$ nT plus an external term $q_1^0 = 20$ nT, B_r and B_θ at $r = a$ with Gaussian noise $\sigma = 5$ nT, least squares) recovers $g_1^0 = -29400.01$ and the external term 20.20, against true values of -29400 and 20 (note 14). The analysis rests on Laplace's equation (MSC 31) and on the split of the field into internal and external parts, with the external part small.

The age of the Earth, 1862–1956

In 1862–63 William Thomson, later Lord Kelvin, estimated the age of the Earth from the conductive cooling of a half-space [16, 17]. The temperature profile is $T = T_0\,\mathrm{erf}(z/2\sqrt{\kappa t})$, so the age is $t = T_0^2/(\pi\kappa G^2)$, where G is the surface gradient. His bounds were 20 million years (Wikipedia) or 24 million (GSA Today) up to 400 million, with a best value of 96 million (GSA Today); in 1897 he gave 20–40 million (Wikipedia) (note 5). A check with $T_0 = 3900$ °C, $\kappa = 1.2\times10^{-6}$ m²/s and a year of 365.25 days: 96 million years needs $G = 36.49$ K/km, while GSA's "about 36 °C/km" gives 98.6 million. A gradient of 20 K/km gives 319.6 million years, and 50 K/km gives 51.1 million. The argument rests on the heat equation (MSC 35) for a solid cooling from a uniform initial temperature. Whether radioactivity refuted it is disputed (note 6).

In 1895 John Perry showed that the Earth can be far older if its interior conducts or convects heat much better than its crust [16, 17]. He used a two-layer model of conductivity, 32 years after Kelvin's paper of 1863. Perry exposed a gap in Kelvin's assumption, not in his arithmetic.

In 1956 Clair Patterson found the age of the Earth to be 4.55 ± 0.07 billion years from meteorites, by a lead–lead isochron [17, 33]. This lies outside the conduction chain, and the value is 47.4 times Kelvin's 96 million years. It replaces Kelvin's age by an independent method.

Tides, oceans and seismic inversion, 1867–1921

From 1867 to 1872 Kelvin put the harmonic analysis of tides on a practical basis [18, 19]. He expanded the tide into constituents with astronomical frequencies and summed them mechanically; the first tide-predicting machine followed in 1872. NOAA says that Laplace had recognised “partial tides” and that Kelvin devised harmonic analysis around 1867. No source says that Kelvin used Laplace’s idea, so the link is inferred from the content.

In 1902–1905 Vagn Walfrid Ekman showed that a wind-driven current is deflected 45° from the wind at the surface [20, 21]. A balance of Coriolis force and friction gives u + iv ∝ e^((1+i)z/D). A sympy check: W = e^((1+i)z/D) satisfies νW″ − ifW = 0 exactly for D = √(2ν/f). Stewart’s Ekman depth is π·D, and empirically $7.6U_{10}/\sqrt{(\sin\varphi)}$ gives 90.4 m for a wind of 10 m/s at latitude 45° (Stewart’s table: about 90 m). The problem was posed by Fridtjof Nansen’s observation of the drift of ice.

In 1907 Gustav Herglotz inverted seismic travel times for velocity as a function of depth by solving an Abel integral equation [22, 23]. A check with the flat-Earth Herglotz–Wiechert formula for a synthetic velocity v = 5 + 0.05z km/s: from the exact $X(p) = 2\sqrt{(1-p^2v_0^2)}/(0.05p)$, the inversion $z(v_1) = (1/\pi)\int_0^{X_1} \operatorname{arcosh}(p(x)/p_1)\,dx$, by adaptive quadrature, returns depths of 100.000 and 150.000 km for v_1 = 10 and 12.5 km/s (true 100 and 150). The method rests on Abel integral equations (MSC 45). Who should be credited is disputed (note 4).

In 1909 Augustus Love introduced the numbers h and k for the elastic yielding of the Earth to tides, and in 1912 Toshi Shida added l [24, 25, 16, 26]. The model is an elastic sphere under a tidal potential of degree 2. A check: with the fluid Love number k_f = 3/2 and m as above, f = (1 + k_f)m/2 = 5m/4 = 0.0043347, exactly Newton’s value. For a homogeneous incompressible sphere, $h_2 = (5/2)/(1 + 19\mu/2\rho gR)$ and $k_2 = (3/2)/(1 + 19\mu/2\rho gR)$, so $h_2/k_2 = 5/3$ for any rigidity μ. The theory builds on the Clairaut–Stokes theory of the equilibrium figure, as this check suggests, though no source consulted names that link (note 12). It also builds on Kelvin’s estimate of around 1863 from the fortnightly tides that the Earth is “as rigid as steel” (GSA Today, Špičáková).

In 1921 Arthur Doodson developed the tide-generating potential harmonically into 388 frequencies, now called Doodson numbers, using E. W. Brown’s lunar theory [19, 29, 30]. This came 54 years after Kelvin’s start in 1867. It builds on the harmonic developments of Kelvin and George Darwin; this link is stated only on two Wikipedia pages, and NOAA and MIT do not name Doodson (note 12).

Dynamo theory, 1919–1966

In 1919 Joseph Larmor proposed the self-exciting dynamo in a report to the British Association [27, 28]. This was a proposal, not a theorem, and it has no earlier link in the chain.

In 1933–34 Thomas Cowling proved that no axisymmetric field can be maintained by dynamo action [27, 28]. The poloidal flux has no source: advection only redistributes it, and resistivity makes it decay (Rincon). The theorem came 95 years after Gauss 1839 (note 2). Larmor’s hypothesis posed the problem, and Cowling ruled out its simplest version.

In 1954 Edward Bullard and H. Gellman truncated the kinematic induction equation

in spherical harmonics [31, 32, 43]. The dynamo solutions they claimed later proved spurious: Gubbins and colleagues note that they "stopped at L = 4" and that the solutions were "later discovered to be bogus" [32]. Bullard and Gellman also proved that there are no dynamos from motions without radial components: "There are no solutions if the motion is purely toroidal" [43]. The work rests on Larmor, who posed the problem (note 8).

In 1956–57 Yakov Zeldovich proved that planar, two-dimensional motions cannot excite a dynamo [28, 34]. Rincon gives 1956 and Wikipedia 1957 (note 3); neither source gives the proof idea. The theorem is analogous to Cowling's, but no use of Cowling's proof is claimed.

In 1958 Rudolf Herzenberg and George Backus gave the first rigorous proofs that homogeneous kinematic dynamos exist [34, 32, 36]. Herzenberg's was a steady dynamo of two eddies. Backus's paper was "self-sustaining dissipative spherical dynamos" (Ann. Phys. 4, 372–447). Backus also gave the necessary bound $R \geq \pi^2$ on a magnetic Reynolds number based on strain; this is known here as Kaiser and Tilgner cite it. These results came 24 years after Cowling 1934, and they answer the challenge of Cowling's theorem with existence.

In 1966 Max Steenbeck, Fritz Krause and Karl-Heinz Rädler founded mean-field electrodynamics [27, 34]. Averaging the induction equation over turbulence, they found that helical turbulence gives an electromotive force proportional to $\alpha\langle B\rangle$, the α effect. This rests on Eugene Parker's cyclonic convection of 1955.

Inversion, simulation and experiment, 1968–2026

In 1968 George Backus and Freeman Gilbert analysed the resolving power of gross Earth data [37, 38, 39]. They built averaging kernels from the data kernels and minimised their spread under a unimodularity constraint, trading resolution against variance. A toy check (Gaussian data kernels with standard deviation 0.15 centred at 0.2, 0.4, 0.6 and 0.8 on the interval from 0 to 1, a 2001-point trapezoid grid, target $x_0 = 0.5$ and spread $12\int(x - x_0)^2A^2$): $\int A = 1.000000$, the peak is at 0.5 and the spread is 0.3006. The method rests on normal-mode data of the elastic Earth; the paper's own examples are mass, moment of inertia, mode frequencies and Q. It continues the line of Herglotz's inversion, but no source consulted links the two (note 12).

In 1995 Gary Glatzmaier and Paul Roberts, and independently Akira Kageyama and colleagues, built self-consistent three-dimensional MHD models of the geodynamo [40, 31]. These reproduced "salient features" of the Earth's field. They are numerical results, not proofs (note 7). There is no arrow from Cowling and Zeldovich: neither source consulted links the models to the antidynamo theorems.

In 1999 the first laboratory dynamos in liquid sodium ran at Riga (Agris Gailitis and colleagues) and Karlsruhe (Ulrich Müller and Robert Stieglitz), with papers in 2000–01 [27, 34]. They were designed on the Ponomarenko and Roberts flows (Rincon), and the Karlsruhe dynamo behaves as an α^2 mean-field dynamo (Rädler). These are experiments, not proofs (note 9). Their design rests on the Roberts and Ponomarenko flows and on mean-field theory.

In 2025 Ralf Kaiser and Andreas Tilgner posted a preprint (arXiv 2510.05884, version 1 of 7 October 2025, with no journal reference as of 29 October 2026) [35, 36]. It proves radially weighted lower bounds of Backus and Childress type on the magnetic

Reynolds number for spherical dynamos, by an energy inequality with a radial weight. For the most efficient flow the Backus-type ratio improves from about 16.3 (abstract; 16.6 = 204/12.3 in section 4 and Table 1) to about 6.4 (note 10). A check: 16.3/6.4 = 2.55 times sharper (2.59 with 16.6), and 204/12.3 = 16.59; the paper came 67 years after 1958. It builds on Backus 1958 ($R \geq \pi^2$, with the maximum strain) and Childress 1969 ($R \geq \pi$, with the maximum velocity), which Proctor improved in 1977 to 12.29 and 3.50, according to the paper itself [36].

In 2026 Ali Arslan and Hezekiah Grayer posted a preprint (arXiv 2608.28584, version 1 of 28 August 2026, with no journal reference as of 29 October 2026) [41]. It derives necessary conditions for instantaneous dynamo action in inertialess rotating convection. The method uses L^p bounds on the flow and a poloidal–toroidal decomposition, and one branch of the constraints implies Ra ≳ Ek^(−3/2). It stands, in the abstract's words, "in contrast to kinematic dynamo bounds", since the magnetic field is coupled to the flow through an inertialess force balance.

Notes

1. The ancient sources give Eratosthenes both 250,000 and 252,000 stadia. With a stadion of 157.2 or 166.7 m the result ranges from 39,300 to 41,675 km, which cannot be settled [1, 2].
2. Cowling's theorem is dated 1933 (Rincon, and the bibcode on Wikipedia) and 1934 (MNRAS 94, in Rädler and Wikipedia). It was probably read in 1933 and printed in the 1934 volume [34, 27, 28].
3. Zeldovich's planar theorem is dated 1956 by Rincon and 1957 by Wikipedia; this has not been settled from the primary source [34, 28].
4. MacTutor credits Herglotz alone with the travel-time inversion, while Wikipedia credits Herglotz together with Wiechert. This attribution conflict has not been settled [22, 23].
5. Kelvin's age is given as 1862 with 20–400 million years (Wikipedia) and as 1863 with 96 million and 24–400 million years (England and colleagues, GSA Today). This is probably one paper and its printing, with the best value depending on rounding [17, 16].
6. Wikipedia says Kelvin failed by ignoring radioactive heat. England and colleagues say that radioactivity did not invalidate the calculation and that convection, as Perry argued, did. This is a conflict of interpretation [17, 16].
7. Wikipedia calls the 1995 models the first self-consistent dynamos, while Deguen and Lasbleis list numerical dynamos from the 1980s (Glatzmaier 1984–85, Zhang and Busse 1988) [40, 31].
8. Bullard and Gellman 1954 and Bullard 1955 are two different papers. Rädler's citation of Parker 1955 contains an error inside that source and is not used [27].
9. Rincon dates the Riga and Karlsruhe results by their papers (2000, 2001); the dating here follows the experiments (1999) [34].
10. Kaiser and Tilgner give the unweighted Backus-type ratio as 16.3 in the abstract and 16.6 in the body (section 4, Table 1), a discrepancy inside the paper [36].
11. Not conflicts: Stewart's Ekman depth differs from the e-folding depth by a factor π by definition, and Ekman 1902/1905 and Laplace 1775–76 against the *Mécanique céleste* are two stages of one work [21].
12. Some statements rest on one source. Laplace's f = 1/330 from 15 gravity values is stated only by Wikipedia [3]; a history of the figure of the Earth credits Laplace with 1/150 from the French meridian arc instead [44]. The link from Kelvin and

Darwin to Doodson rests on Wikipedia. Only Kaiser and Tilgner's paper gives the Backus and Childress bounds and Proctor's improvement [36]. The links from Clairaut and Stokes to Love, and from Herglotz to Backus and Gilbert, are inferred from the content; no source consulted states them.

13. The date of Eratosthenes rests on one cited book [1, 2].
14. The geomagnetic check is a random toy; its recovered values depend on the order of the random draws, but in every variant they agree with the true values within the noise.
15. Not covered: the Taylor–Proudman theorem, Rossby waves, the mathematics of plate tectonics, Braginsky 1964, Roberts 1970/72 and Ponomarenko 1973 as separate links, and Lorenz 1963. Apart from the two preprints, no rigorous geophysics result from 2025–26 was found. Elias-López's thesis "Numerical simulations of astrophysical dynamos and applications to giant planets" (arXiv 2512.09725, version 1 of 10 December 2025, no journal reference) is numerical work without proofs, so it adds no rigorous 2025–26 result [42].

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Operations research

Operations research, in the MSC2020 class “Operations research, mathematical programming”, studies how to choose the best action under constraints [41, 42]. Its chain starts with Monge’s transport problem of 1781 and Fourier’s inequalities of 1826–27. Farkas’s theorem of alternatives of 1902 then carries a line of duality through Karush, von Neumann’s conjecture, Gale–Kuhn–Tucker and Kuhn–Tucker. Kantorovich turned Monge’s problem into linear programming in 1942, and Ford and Fulkerson answered Harris’s problem of maximum flow. The simplex method (see the chapter on numerical mathematics) opens a line on complexity, from Klee–Minty through Khachiyan and Spielman–Teng to the optimal smoothed bound of Bach and Huiberts (arXiv, revised May 2026). Approximation of the travelling salesman problem moves from the factor 3/2 of Christofides and Serdyukov (1976) to Karlin, Klein and Oveis Gharan (2020).

Three pairs have no arrow between them. Tolstoĭ and Kantorovich worked in parallel. Kuhn and Tucker rediscovered Karush’s conditions. Christofides and Serdyukov found the same algorithm independently. The chain has 21 links, from Monge to Bach and colleagues.

Transport and inequalities, 1781–1942

In 1781 Gaspard Monge posed the problem of *déblais et remblais*: move one distribution of mass onto another by a map, at least cost [1, 2]. A check with scipy’s linprog (HiGHS) and cost $|x - y|$: moving δ_0 to $\tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_1$ has no Monge map, since a map sends the point 0 to a single point. The Kantorovich linear program, however, gives the plan (0.5, 0.5) at cost 1.0. Monge’s problem has no earlier link in this chain.

In 1826–27 Joseph Fourier gave a method for systems of linear inequalities: eliminate the variables one by one by pairing upper and lower bounds [3, 4, 5, 6]. The date is disputed (note 2). This link too stands at the start of its line.

In 1902 Gyula Farkas proved his theorem of alternatives in “Theorie der einfachen Ungleichungen” [3, 7, 5]. Either $Ax = b$ with $x \geq 0$ has a solution, or some y has $A^{\mathrm{T}}y \geq 0$ and $b^{\mathrm{T}}y < 0$. None of the sources consulted gives his method of proof. A check with linprog: for $A = [[1, 1], [1, -1]]$ and $b = (-1, 0)$ the system is reported infeasible, and $y = (1, 0)$ gives $A^{\mathrm{T}}y = (1, 1) \geq 0$ and $b \cdot y = -1 < 0$. Fourier’s method was an influence only: MacTutor speaks of “the nearly forgotten Fourier principle”, and Godard says Farkas was “inspired by Fourier”.

In 1930 A. N. Tolstoĭ stated a cycle criterion: an optimal transport has no cycle of negative cost in its residual graph [8, 9]. He assumed sufficiency without proof. His instance of 10×68 rail connections was later confirmed optimal by the CPLEX solver. Only Schrijver describes this work (note 7).

In 1939 Leonid Kantorovich formulated production planning as linear programming [10, 11, 12]. His “resolving multipliers”, which are dual prices, certify optimality. The work dates from July 1939 and has no earlier link in this chain.

Also in 1939 William Karush found necessary conditions for minima under inequality constraints, in a master's thesis that remained unpublished until Kuhn's talk of 1975 [13, 14]. According to Kjeldsen it rests on Farkas's lemma, while Wikipedia's page on the conditions cites the separation of hyperplanes instead (note 7).

In 1942 Kantorovich relaxed Monge's problem [1, 2, 15, 12]. He optimised over transport plans, which he called "translocations"; this allows mass to be split and makes the problem linear. A plan is minimal if and only if it is "potential", which is Kantorovich duality. A check on a small instance (supplies 50 and 70, demands 30, 40 and 50, costs [[2, 4, 5], [3, 1, 7]], primal and dual solved by linprog): the primal minimum 380.0 equals the dual maximum 380.0. The note of 1942 does not cite Monge; Kantorovich linked the two problems in his note of 1948, "On a problem of Monge" (Vershik). It also builds on his multipliers of 1939, in which "the idea of duality was contained from the very beginning" (Vershik, MacTutor). There is no arrow from Tolstoĭ: their work was parallel.

Duality and nonlinear programming, 1947–1951

In 1947 George Dantzig stated linear programs in July and found the simplex method by the end of that summer [16, 17]. The full chain is told in the chapter on numerical mathematics.

In November 1947 John von Neumann tied the linear maximum problem to zero-sum games in a note, and conjectured the converse [13, 18]. He used an incorrect form of Farkas's lemma. Kjeldsen says it is unclear whether the note contains dual variables. The work rests on the minimax theorem and the equivalence of linear programs and games (MSC 91).

In 1951 David Gale, Harold Kuhn and Albert Tucker proved the duality theorem of linear programming and the equivalence with games, in "Linear programming and the theory of games" [13, 19, 7]. Farkas's lemma is now usually stated in their form. None of the sources consulted gives their method of proof. A check: the instance of Kantorovich 1942 above has primal = dual = 380. The theorem proves von Neumann's conjecture.

In 1950–51 Kuhn and Tucker stated the Kuhn–Tucker conditions for nonlinear programming [13, 14, 19, 20]. They used a saddle point of the Lagrangian, which extends duality in linear programming. The talk was given at Berkeley in 1950 and the proceedings appeared in 1951. The work rests on LP duality, which Kuhn cast as a Lagrangian saddle point (Kjeldsen; Kuhn 1976). According to Kjeldsen it also rests on Farkas's lemma (note 7). There is no arrow from Karush: Kuhn and Tucker rediscovered his conditions.

Networks, 1954–1956

On 24 October 1955 Theodore Harris and Frank Ross posed the problem of maximum flow on the rail network of the Soviet Union and Eastern Europe, in RAND report RM-1573 [8, 21]. They were also interested in the minimum cut. Their maximum of 163,000 tons a day is limited by a bottleneck cut, drawn as the line X–Y; the report does not state the value of the cut separately (note 3).

In 1954–56 Lester Ford and Delbert Fulkerson proved the max-flow min-cut theorem,

which they called the “minimal cut theorem” [8, 22]. Flows form a closed convex polytope, so a maximal flow exists. The arcs saturated in every maximal flow contain a disconnecting set that each chain crosses once, and its capacity equals the flow (Can. J. Math. 8). A check on a small network (s→a 10, s→b 5, a→b 15, a→t 9, b→t 10), solved by breadth-first augmenting paths: the maximum flow 15 equals the minimum cut 15, the cut around {s}. The theorem rests on Harris’s problem: the 1956 paper, received on 20 September 1955, says the problem was “formulated by T. Harris” (note 1).

Complexity of linear programming, 1972–2026

In 1972 Victor Klee and George Minty showed that the simplex method with Dantzig’s rule visits all 2^D vertices of a deformed cube, taking $2^D - 1$ pivots [23, 24, 25]. A check with a dense-tableau simplex using the most negative reduced cost, on the Klee–Minty form “maximise $\Sigma\ 2^{n-j}x_j$ subject to $\Sigma_{(j<i)}\ 2^{(i-j+1)}x_j + x_i \le 5^i$”: it takes 3, 7, …, 255 $= 2^n - 1$ pivots for n = 2 to 8, with optimum 5^n. The result is a counterexample to Dantzig’s rule.

In 1979 Leonid Khachiyan proved that linear programming is solvable in polynomial time by the ellipsoid method, whose ellipsoids shrink in volume at every step [28, 29, 24, 30]. His four-page note in Soviet Mathematics Doklady (February 1979) had no proofs; they followed in 1980. The question had been posed by Klee–Minty (Wikipedia, Todd). The method rests on the ellipsoid method of Yudin and Nemirovski and, independently, Shor, in 1976–77 (Todd, Grötschel) (note 5).

In 2001 Daniel Spielman and Shang-Hua Teng introduced smoothed analysis, published in J. ACM 51 in 2004 [31, 32, 33]. They proved that the simplex method has polynomial smoothed complexity under small random perturbations of the input, with a bound of $O(\sigma^{-30}\ d^{55}\ n^{86})$, as quoted by Bach and Huiberts. A check at $\sigma = 0.01$, d = 100 and $n = 10^4$: $\log_{10}$ of the bound is 60 + 110 + 344 = 514. The simplex method and Klee–Minty posed the problem.

In 2010 Francisco Santos disproved the Hirsch conjecture, which Hirsch had stated in a letter to Dantzig in 1957 [34, 35, 36]. He built a 43-dimensional polytope with 86 facets from a 5-dimensional one with 48 facets. A check: n − d = 43, while the diameter is at least 44. The result is a counterexample, as both the paper and Wikipedia describe it.

In 2025 Eleon Bach and Sophie Huiberts gave an optimal smoothed analysis [33, 39]. A new variant of the simplex method takes $O(\sigma^{(-1/2)}\ d^{(11/4)}\ \log(n)^{(7/4)})$ pivots, and a lower bound shows that the dependence on the noise is optimal up to polylogarithmic factors. A check with $\sigma = 0.01$, d = 100, $n = 10^4$ and the natural logarithm inside: $\log_{10}$ of the bound is 8.19. This compares with 11.19 for Huiberts, Lee and Zhang 2023 ($O(\sigma^{(-3/2)}\ d^{(13/4)}\ \log(n)^{(7/4)})$) and 514 for Spielman–Teng. The work builds on the framework of Spielman and Teng. It remained an arXiv preprint (version 2 of 23 May 2026) with no journal reference as of 29 October 2026.

In 2025 Bach, Alexander Black, Huiberts and Sean Kafer gave a “by the book” analysis: the simplex method is polynomial once the implementation, with its scaling and tolerances, is modelled together with the input [40]. The paper, “Beyond Smoothed Analysis” (arXiv version 2 of 26 June 2026), presents this as going beyond Spielman and Teng’s smoothed analysis, which models the input data only (note 7).

Travelling salesman, 1976–2020

In 1976 Nicos Christofides gave a 3/2-approximation for the metric travelling salesman problem [26, 27]. Take a minimum spanning tree, add a minimum perfect matching on the vertices of odd degree, and shortcut an Euler tour. Anatoliy Serdyukov found the same algorithm in 1976 and published it in 1978 (note 4). There is no arrow between them: the discoveries were independent.

In 2020 Anna Karlin, Nathan Klein and Shayan Oveis Gharan gave a $(3/2 - \varepsilon)$-approximation for the metric problem with $\varepsilon > 10^{-36}$ [37, 38]. They replaced the minimum spanning tree by a random spanning tree drawn from a designed distribution, and the paper won the best paper award at STOC 2021. A check in Python: $3/2 - 10^{-36} < 3/2$ holds for exact fractions, but in double precision the difference vanishes and the value equals 1.5. The work rests on Christofides’s template of tree and matching.

Notes

1. Ford and Fulkerson (1954) say Harris formulated the max-flow problem, while their 1962 book says he posed it in spring 1955; Schrijver notes the discrepancy [8]. The primary texts do not settle it: the 1956 paper, received on 20 September 1955, says “formulated by T. Harris”, and the Harris–Ross report is dated 24 October 1955 [22, 21].
2. Fourier’s method is dated 1826 by Wikipedia and by Godard, who lists papers of 1823, 1824 and 1826, but 1827 by Dahl. This has not been settled [6, 5, 4].
3. Schrijver says Harris and Ross found a cut of capacity “again, 163,000 tons”, while the report’s text states only the flow and draws the bottleneck X–Y [8, 21]. The value of the cut is therefore Schrijver’s reading. The 1956 proof of Ford and Fulkerson goes through saturated arcs and a polytope, not through augmenting paths [22].
4. The 3/2 algorithm is named after Christofides (1976), but Serdyukov found it in 1976 and published it in 1978 [26].
5. The ellipsoid method is dated 1976–77, to Yudin and Nemirovski and independently Shor, by Todd [24]. Wikipedia’s date of 1972 has no support from a second source [28].
6. Kuhn and Tucker’s work is dated 1950 for the Berkeley talk and 1951 for the proceedings. Kantorovich’s early work dates from July 1939.
7. Some statements rest on one source. Only Schrijver describes Tolstoĭ’s work [8, 9]. Only Kjeldsen states that Karush and Kuhn–Tucker rest on Farkas’s lemma [13]. Only the “by the book” paper itself states its relation to smoothed analysis; no second source was found [40].
8. Open as of 29 October 2026: strongly polynomial linear programming and the polynomial Hirsch conjecture [43, 44, 36]. Bellman’s dynamic programming (1957) and Gomory’s cutting planes (1958) lie outside this chain. The MSC2020 title is confirmed by the AMS [41].

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Game theory and economics

Game theory studies decisions whose outcome depends on what others decide; mathematical economics, social choice and fair division grow from the same root. The chain starts with the Talmud's division of an estate and Condorcet's 1785 voting cycle. Its spine runs from Zermelo's chess theorem (1913) through von Neumann's minimax theorem (1928) and the book of von Neumann and Morgenstern (1944) to Nash's equilibrium proofs (1950/51), which rest on the fixed-point theorems of Kakutani and Brouwer.

From Nash the chain branches in two directions: to general equilibrium (Debreu 1952, Arrow–Debreu 1954) and to computational hardness (the class PPAD, 2005–2006). Social choice runs from Condorcet through Arrow (1951) and Gibbard–Satterthwaite (1973/75) to a topological model of 2026. Fair division runs from Steinhaus (1944) to a counterexample on EFX allocations in April 2026. As of 28 October 2026 both 2026 results are preprints, and EFX for additive valuations with $n \geq 4$ agents is still open.

Two apparent links are not links. Arrow–Debreu, McKenzie (1954) and Gale (1955) proved their equilibrium theorems independently. Bachelier influenced the later option-pricing theory, but Black and Scholes did not build on his proof.

Old problems: the Talmud, Condorcet, Cournot and Bachelier

The oldest link is a passage of the Talmud (Ketubot 93a). The Mishna of Rabbi Nathan divides an estate of 100, 200 or 300 among three claims of 100, 200 and 300 [1, 2]. The Mishna was fixed about 200 CE, some 1800 years before Gura and Maschler wrote about it. The Gemara followed about 200 years later, and final editing of the Babylonian Talmud ran to about 500 CE [2, 3]. The passage rests on no earlier mathematics (note 10).

In 1785 Condorcet showed that pairwise majority voting can cycle. Three voters with cyclic rankings suffice: A beats B, B beats C and C beats A, each by a majority of 2/3 (a direct count confirms this) [4, 5].

In 1838 Cournot described an equilibrium among oligopolists: each firm chooses its best quantity given the quantities of the others [6, 7]. This was a strategic analysis of one specific situation, with no general method.

In 1900 Bachelier's thesis *Théorie de la spéculation* priced options with a Brownian model of prices. It also gave the distribution of the Wiener process, five years before Einstein [8, 9]. A quick comparison shows how close his formula comes to the later standard. At the money, with $S = K = 100$, $r = 0$, $T = 1$ and an absolute volatility of 20, Bachelier's price is $20/\sqrt{(2\pi)} = 7.9788$. Black–Scholes with $r = 0$ and $\sigma = 20\%$ gives 7.9656.

From chess to minimax: Zermelo, König, Kalmár, Borel and von Neumann

In 1913 Zermelo proved that a player who can force a win in chess can do so in fewer moves than there are positions [10, 11]. He left open whether the starting position is a win. The familiar statement "win, loss or forced draw" is later framing, as Schwalbe and Walker point out [10]. Zermelo argued by contradiction from the fact that positions need not repeat; he did not use backward induction. The same idea settles simple games such as Nim: the position (1, 2, 3) is a loss for the first player, since 1 xor 2 xor 3 = 0, while (1, 2, 4) and (3, 4, 5) are wins, with xor 7 and 2.

Between 1921 and 1927 Borel defined games of strategy in a series of notes [12, 13]. They rest on no earlier result in this chain.

In 1927 König showed that a win can be forced within a bounded number of moves even when there are infinitely many positions [10, 11, 14, 15]. His proof uses his lemma on infinite paths in finitely branching graphs. This builds on Zermelo 1913. König pointed out a gap in Zermelo's argument; Zermelo answered with a new proof in an appendix, and Kalmár later closed the gap [10, 14] (note 2). According to Schwalbe and Walker, König's bound also answered a conjecture of von Neumann (note 11).

In 1928 von Neumann proved the minimax theorem for two-person zero-sum games [6, 16]. With mixed strategies p and q and payoff matrix A, the maximum over p of the minimum over q of $p^{T}Aq$ equals the minimum over q of the maximum over p. A linear-programming check confirms the standard examples. Rock-paper-scissors has value 0 with strategy (⅓, ⅓, ⅓), and matching pennies has value 0 with (½, ½). The 3×3 matrix A = [[1,0,1],[0,1,0],[1,0,0]] has maxmin = minmax = 0.5. MacTutor says that Borel's work on games led von Neumann to the subject, so the theorem was influenced by Borel [12, 13] (note 4).

In 1928/29 Kalmár extended the result to infinitely many positions and infinitely many successors [10, 11]. He showed that a possible win can be enforced without any position repeating. This generalises König 1927 and repairs the gap König had found.

Equilibrium: von Neumann–Morgenstern, Nash, Debreu, Shapley and Arrow–Debreu

In 1944 von Neumann and Morgenstern published *Theory of Games and Economic Behavior* [6, 16]. It contains the theory of zero-sum games, a cooperative theory for n players and, according to Schwalbe and Walker, backward induction [10] (note 1). The book rests on von Neumann 1928: the minimax theorem is its "main theorem" for two-person zero-sum games, and Nash's 1950 paper cites it as Chapter 3 [6, 21].

Nash proved in 1949/50 that every finite game with n players has an equilibrium point (PNAS 36: 48–49) [21, 22]. The proof shows that the best-reply correspondence has a fixed point. The paper was communicated on 16 November 1949 and published on 15 January 1950, 60 days later (note 3). It rests on Kakutani's fixed-point theorem, which Gale suggested to him, and it generalises the zero-sum case of von Neumann and Morgenstern. A small example shows a mixed equilibrium. In the battle of the sexes, with payoffs (2, 1) and (1, 2) on the diagonal and 0 elsewhere, the equilibrium is p = 2/3, q = 1/3. Cournot's 1838 equilibrium is a special case, but there is no arrow from it: neither Nash paper names Cournot, and the link is one of influence only [6].

In 1951 Arrow proved his impossibility theorem [4, 5]. A rule that satisfies the Pareto condition, independence of irrelevant alternatives and universal domain must have a dictator. The proof works through decisive coalitions, which form an ultrafilter and so contain a dictator, as Grossi's notes show [5]. The theorem generalises Condorcet 1785. Majority voting meets every one of Arrow's conditions except ordering, which fails exactly at Condorcet's cycle [4]. Grossi notes that the proof's profile resembles the paradox [5]. Neither source says that the proof uses Condorcet's argument, so the relation is one of content rather than of method.

Nash's second existence proof appeared in 1951 in the Annals of Mathematics 54 [22, 7]. It uses a continuous improvement map on profiles of mixed strategies, with Brouwer's fixed-point theorem in place of Kakutani's. It rests on Nash 1950 and on Brouwer's theorem.

In 1952 Debreu proved that an equilibrium exists in an abstract economy [23, 24]. His theorem contains Nash's, with convexity replaced by acyclicity. It rests on Nash 1950, which Debreu cites; Arrow and Debreu later call the lemma a generalisation of Nash's theorem [24]. Debreu also presents the result as a particular case of the Eilenberg–Montgomery fixed-point theorem (note 12).

In 1953 Shapley proved that one value satisfies efficiency, symmetry, additivity and the null-player property, and that it is unique [25, 26]. It is the average marginal contribution of a player over all orderings of the players (note 5). In the glove game, player 1 holds a left glove and players 2 and 3 hold right gloves, and a pair is worth 1. The exact average over the 6 orderings gives (2/3, 1/6, 1/6).

In 1954 Arrow and Debreu proved that a competitive equilibrium exists (Econometrica 22) [24, 27, 28]. They add a market participant as a player, which turns the economy into an abstract economy; Debreu's lemma then gives the equilibrium. The proof rests on Debreu 1952, cited as their reference [5]: their lemma "is a special case of the Theorem in [5]" (note 12). From Debreu's PNAS paper (communicated 1 August 1952) to Econometrica 22(3), the July 1954 issue, is 699 days. McKenzie (1954) and Gale (1955) proved similar existence theorems independently, so there is no arrow between them and Arrow–Debreu [27, 28].

Matching, social choice and option pricing

The fair-division problem was posed by Steinhaus in 1944 [17]. He presented it to the Econometric Society in 1947 and published it in *Econometrica* 16 (1948) 101–104 [18, 20] (note 7). He gave a division for three people that is proportional but not envy-free [17, 19]. It rests on no earlier link.

Around 1960 Selfridge and Conway independently found an envy-free division for three people [29, 30, 20, 31] (note 6). One player trims a piece, and a second round divides the trimmings. The protocol rests on the problem Steinhaus had posed in the 1940s, and on the notion of envy-freeness from Gamow and Stern (1958), whom Aziz and Mackenzie cite for the envy-free problem [20, 31].

In 1962 Gale and Shapley proved that stable marriages always exist [32]. Their deferred-acceptance algorithm finds the stable matching that is best for the proposing side, in at most $n^2 - 2n + 2$ stages. Simulations confirm both claims. No run produces a blocking pair. All 46,656 preference profiles for n = 3 give a maximum of 5 stages, equal to the bound. For n = 2 to 6, 20,000 random profiles each (numpy default_rng,

seed 0) never exceed the bound; the maxima are 2, 5, 10, 13 and 20, against bounds 2, 5, 10, 17 and 26. The theorem rests on no earlier link. The clearinghouse of the US National Resident Matching Program, set up in the 1950s, was later found to be closely related to deferred acceptance (Roth 1984), and its 1997 redesign by Roth and Peranson built on it [33, 34].

Gibbard (1973) and Satterthwaite (1975) proved that every non-dictatorial voting rule with at least three outcomes can be manipulated [35, 36, 4]. A strategy-proof rule would yield a social welfare function meeting Arrow's other conditions, which Arrow's theorem forbids. The result rests on Arrow 1951 and settles a conjecture of Dummett and Farquharson (1961), quoted in Endriss's slides [36].

In 1973 Black and Scholes, and Merton, gave the option-pricing formula [8, 9]. A riskless hedge, rebalanced continuously, must earn the risk-free rate, so no risk premium is needed. For a call with S = K = 100, r = 5%, σ = 20% and T = 1, the formula gives 10.4506. A risk-neutral Monte Carlo run with 10^6 paths (numpy default_rng, seed 0) gives 10.4697 ± 0.0147. A daily delta hedge (252 steps, 10,000 paths, seed 1) under an 8% drift has a mean error of 0.0055, with standard deviation 0.4294. There is no arrow from Bachelier 1900. The Nobel Prize committee calls his formula flawed, and MacTutor says Black and Scholes followed Osborne and Samuelson, so the link is one of influence only [8, 9].

In 1985 Aumann and Maschler showed that the Talmud's numbers are exactly the nucleoli of the corresponding bankruptcy games [1, 2]. The rule behind them is the contested-garment principle. This rests on the Talmud passage of c. 500 CE. Two computations agree: the consistent rule and an exhaustive nucleolus search on a grid of ⅓ units both give (33⅓, 33⅓, 33⅓), (50, 75, 75) and (50, 100, 150).

Computation and recent fair division

In 1991 Papadimitriou defined PPAD, the class of problems solved by a polynomial parity argument on directed graphs; the journal paper followed in 1994 [37, 38] (note 8). Brouwer's fixed-point problem is complete for this class. The definition rests on no earlier link in this chain.

In 1995 Brams and Taylor gave an envy-free protocol for any number of players, but its running time is unbounded [29, 39, 40]. It uses the idea later named domination. It rests on no earlier link listed here.

In 2005 Daskalakis, Goldberg and Papadimitriou proved that finding a Nash equilibrium for 4 or more players is PPAD-complete; the 2009 SICOMP version covers 3 players [37, 41]. Graphical games are used to encode Brouwer functions. This rests on Papadimitriou 1991, and on Nash 1950/51, whose non-constructive existence proof posed the problem.

In 2006 Chen and Deng proved that two-player Nash equilibrium is PPAD-complete, and with Teng (arXiv 2007) that it has no FPTAS unless PPAD ⊆ P [38, 37]. As a consequence, Arrow–Debreu equilibria are PPAD-hard, as the Chen–Deng–Teng abstract states. The result rests on the 2005 work of Daskalakis, Goldberg and Papadimitriou, but the two groups describe the relation differently (note 9).

In 2015 Aziz and Mackenzie gave a discrete, bounded envy-free protocol for 4 agents, and in 2016 for any n, with up to $n^n n^n$n^n queries [29, 20, 51]. This settled a major open question, 55 years after Selfridge and Conway. The bound is very large

even for n = 2: it is $2^{(2}65536$), and 2^65536 alone has 19,729 digits. The protocol rests on Selfridge–Conway and on Brams–Taylor 1995, whose idea of "domination or 'irrevocable advantage' " the paper credits [20, 40].

In 2020 Chaudhury, Garg and Mehlhorn proved that EFX allocations (envy-free up to any item) exist for 3 agents with additive valuations [42, 43, 45, 46, 47]. The paper also disproves a conjecture of Caragiannis et al.: it gives a three-agent instance where a partial EFX allocation has higher Nash welfare than any complete one [42]. The result rests on the EFX notion of Caragiannis et al. (EC '16) [47].

In 2026 Livson, Pritam and Prokopenko showed that Condorcet's paradox is non-orientable [48, 49]. Gluing the rims of the profile space, where the cycles live, gives a Klein bottle or the real projective plane. The paper was submitted to arXiv on 12 January 2026 (v6 on 16 September 2026) and has no journal reference, so it is a preprint as of 28 October 2026. It comes 241 years after Condorcet. It rests on Condorcet 1785 and generalises Baryshnikov's model of 1993; it also cites Chichilnisky's topological social choice, whose footnote Quanta dates to 1980 [48, 49].

Also in 2026 Akrami, Mayorov, Mehlhorn, Srinivas and Weidenbach showed that EFX allocations need not exist for monotone, and hence for submodular, valuations with n ≥ 3 agents and m ≥ n + 5 goods [43, 44]. Every instance with 3 agents and 7 goods does have one. The counterexample was found by SAT solving, with the encoding verified in Lean. The search spaces are small: 3^8 = 6561 allocations of 8 goods to 3 agents, and 3^7 = 2187 of 7. The paper appeared on arXiv on 20 April 2026 (v3 on 14 May 2026) and is a preprint as of 28 October 2026. It refutes the EFX existence conjecture in this setting. Mackenzie and Suzuki then gave a 3-agent, 8-good submodular counterexample (arXiv 2605.06451, May 2026), which Akrami et al. say was motivated by their work [50]. The paper also uses Chaudhury, Garg and Mehlhorn 2020: it attributes its Proposition 2, on non-degenerate counterexamples, partly to them (note 13).

Notes

1. Wikipedia credits Kalmár (1928) with the first use of backward induction. Schwalbe and Walker place it in von Neumann–Morgenstern (1944), and MacQuarrie's MacTutor project says it was not employed until 1953 [10, 11, 15]. Kalmár's 1928 paper itself was not consulted.
2. König's objection, that Zermelo assumed Black never changes behaviour, is reported by Schwalbe and Walker, who judge it incorrect, and by arXiv 1407.1647, which says Zermelo accepted it and corrected his proof [10, 14]. Both versions stand.
3. Nash's first paper is dated from the primary source: written and communicated in 1949, published in 1950 [21].
4. MacTutor credits Borel's minimax paper with leading von Neumann to game theory, while Wikipedia records von Neumann's 1953 note that Borel had rejected the minimax theorem [12, 13, 16]. Both versions stand. Only MacTutor states the influence.
5. The Shapley value is dated 1951 by Wikipedia and 1953 by INFORMS [25, 26]; the earlier date is probably a RAND memorandum, the later one the publication.
6. Selfridge–Conway is dated about 1960 by Aziz–Mackenzie and Quanta, but 1962 by MacTutor [20, 29].

7. Steinhaus proposed the problem in 1944 and published it in Econometrica 16 (1948) 101–104, as Hill cites it; Aziz and Mackenzie add that he presented it to a wider audience on 17 September 1947 [17, 18, 20].
8. PPAD was introduced in 1991 and published in a journal in 1994 [37].
9. Chen–Deng date to FOCS 2006 and the arXiv version with Teng to April 2007; these dates agree. But Daskalakis, Goldberg and Papadimitriou describe Chen and Deng as "improving our construction", while Chen, Deng and Teng describe a direct reduction that bypasses the graphical game model (their §1.4) [37, 38].
10. The Talmud passage is a Mishna, fixed c. 200 CE according to Gura and Maschler, inside the Babylonian Talmud, whose final editing ran to c. 500 CE according to Chabad [2, 3]. Aumann and Maschler's "2000-year-old" is a round figure.
11. Only Schwalbe and Walker state that a von Neumann conjecture lay behind König's 1927 bound [10]. Neither arXiv 1407.1647 nor MacQuarrie's MacTutor chapter mentions it [14, 15].
12. Debreu's use of the Eilenberg–Montgomery theorem, and Arrow and Debreu's use of Debreu 1952, are stated only in the papers themselves [23, 24]. Wikipedia says Arrow and Debreu invoked Kakutani's theorem [28].
13. The link from Akrami et al. to Chaudhury, Garg and Mehlhorn rests only on the paper's own Proposition 2, which it attributes to CGM24 and AAC+25a [44]. No venue for arXiv 2604.18216 is known as of 29 October 2026.
14. Not researched for this chapter: a link from Bernoulli 1738 to the expected utility of von Neumann and Morgenstern (the SEP treats both but states no link), and mathematical finance beyond Bachelier and Black–Scholes (Itô to Merton, Harrison–Kreps, Harrison–Pliska, Delbaen–Schachermayer). The MSC classes covered are 91A Game theory, 91B Mathematical economics, 91C–91F social and behavioural sciences and 91G Actuarial science and mathematical finance (MathSciNet; 91G is missing from the older Belgrade mirror).

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Mathematical biology

Mathematical biology describes living populations, genes, cells and epidemics with equations. The chain starts with Fibonacci's rabbit count (1202) and Malthus's growth argument (1798), which posed the problem that Verhulst's logistic equation (1838) answers.

Three lines follow. The first concerns interacting populations (Lotka, Volterra). The second is population genetics, from Hardy–Weinberg (1908) through Fisher, Haldane and Wright to Kimura (1968) and Kingman's coalescent (1982). The third covers spatial and physiological models (Fisher–KPP, Turing, Hodgkin–Huxley), alongside epidemic thresholds from Ross to Kermack and McKendrick.

In 2026 the Fisher–KPP line still yields new theorems, but the latest link is a single preprint. There is no arrow from Fibonacci to Malthus. Several links are parallel discoveries or rediscoveries rather than dependencies: Lotka and Volterra, Hardy and Weinberg, Fisher and KPP, and the rediscoveries of the logistic curve by McKendrick and by Pearl and Reed.

Growth: Fibonacci, Malthus, Verhulst and the rediscoveries

In 1202 Fibonacci posed the rabbit problem [1, 2, 3]. Mature pairs breed one pair a month, and newborn pairs mature after a month, so $N(n) = N(n-1) + N(n-2)$. Starting from one pregnant pair, there are 377 pairs after 12 months. This is Fibonacci's own answer in the *Liber abaci* text (Boncompagni 1857, p. 283), and the recurrence reproduces it. The rabbit count rests on no earlier link, and it leads to no later one here: Malthus did not build on it (note 1).

In 1798 Malthus stated his principle of population [4, 5, 6]. Unchecked population grows geometrically (1, 2, 4, 8, …), while subsistence grows arithmetically (1, 2, 3, 4, …). This is an argument, not a proof. With doubling every 25 years the ratio is 256 to 9 after two centuries and 4096 to 13 after three. Malthus's own figures are 512 to 10 after two and a quarter centuries and 4096 to 13 after three, which the two sequences confirm [6].

In 1838 Verhulst answered Malthus with the logistic equation $dP/dt = rP(1 - P/K)$ [5, 7]. He added a brake term $-rP^2/K$ to Malthusian growth. He fitted the closed-form S-curve to Belgian data, with a limit of 9.4 million. He named the curve "logistique" in 1844, published in 1845 (note 2). A quick numerical check: with $K = 9.4$ (million), $r = 0.03$ per year and $P_0 = 4$, forward Euler with a step of 0.01 year gives $P(100) = 8.8081$, against 8.8080 in closed form. The equation rests on Malthus 1798, who posed the problem.

McKendrick in 1911, for bacteria, and Pearl and Reed in 1920, for human populations, rediscovered the logistic curve and fitted it to new data [5, 7]. There is no arrow from Verhulst: these were rediscoveries that did not use his work (note 3).

Population genetics: Hardy–Weinberg to Wright

In 1908 Hardy and Weinberg independently proved the law that bears their names [8, 9, 10]. Under random mating, the genotype proportions $p : 2q : r$ satisfy $q^2 = pr$ after one generation and then stay fixed. One round of random mating suffices [10]. For example, starting from (0.5, 0.1, 0.4) one generation gives (0.3025, 0.495, 0.2025), where $q^2 = pr = 0.061256$; the allele frequency 0.55 is unchanged in the second generation. The two discoveries were parallel, with no dependency (note 4).

In 1918 Fisher showed that many Mendelian factors plus the environment give continuous variation and the observed correlations between relatives [10, 13]. This unified biometry and Mendelism. According to the SEP, the paper rests on Hardy–Weinberg 1908 [10]. Modern accounts of the model set it up with genotypes in Hardy–Weinberg equilibrium [14] (note 5).

In 1922 Fisher added selection to the Hardy–Weinberg recursion [18, 19]. He showed that heterozygote advantage keeps both alleles. He also treated the chance that a mutant is lost as a branching process. Bacaër describes the model as “a variant of the Hardy–Weinberg model including natural selection”, so it rests on Hardy–Weinberg 1908 [18]. The branching process itself has an earlier, independent root in Bienaymé (note 6).

In 1927 Haldane showed that a beneficial mutant survives with probability about 2s [20, 21, 22]. With Poisson(1 + s) offspring, the extinction probability q solves q = e^((1+s)(q−1)). Solving this gives a survival probability of 0.01974 for s = 0.01, against 2s = 0.02, and 0.09370 for s = 0.05, against 0.1. The result uses the same branching-process model as Fisher 1922; Patwa and Wahl call it the “Haldane–Fisher” model (note 7). The result appeared in Part V (“Selection and mutation”) of Haldane’s series of 1924–1934 (note 8).

In 1931 Wright described random genetic drift [25, 10]. In a finite population of N individuals, each generation is a binomial resample of the last (the Wright–Fisher model), and one allele is eventually fixed. This rests on Hardy–Weinberg 1908, with the assumption of an infinite population dropped. In Bacaër’s words, the model keeps the Hardy–Weinberg assumptions, but “the population is not assumed infinitely large” [25].

Interacting populations, waves and epidemics

In 1911 Ross showed that malaria persists only if mosquito density exceeds a threshold [11, 12]. He built a mathematical model of transmission, from which it followed that killing a fraction of the mosquitoes is enough. The model rests on no earlier link.

Lotka (1920–1925) and Volterra (1926) wrote down the predator–prey equations $x' = ax − bxy$, $y' = dxy − cy$ [15, 16, 17]. Separating variables gives the invariant $V = dx − c \ln x + by − a \ln y$, so the orbits are closed. Volterra worked independently, from d’Ancona’s data on fish catches. With $a = b = c = d = 1$ and $(x, y) = (2, 1)$ at $t = 0$, $V = 3 − \ln 2 = 2.30685282$, and a high-accuracy integration (scipy DOP853, tolerances 10^{-12}) returns the same value at $t = 20$. The equations build on Verhulst’s logistic, though the sources make this claim separately for Lotka and for Volterra (note 9). Lotka and Volterra worked in parallel, with no dependency.

In 1927 Kermack and McKendrick proved an epidemic threshold and the final-size

equation z = 1 − e^(−R₀z) [23, 24, 12, 11]. The equation has a positive root only above the threshold. The original paper (*Proc. R. Soc. A* 115, 700–721) gives the threshold density $N_0 = 1/A$ and the relation $-\log(1 - p) = ApN$, which is the same equation with $R_0 = AN$ [23]. Solving it gives z = 0 for $R_0 = 0.9$, 0.5828 for $R_0 = 1.5$, 0.7968 for $R_0 = 2$ and 0.9405 for $R_0 = 3$. The method rests on Ross 1911: Aeon says the pair “extended Ross’s method” [11]. The paper itself cites Ross and Hudson (1915–17), “in which the same problem is attacked” [23].

In 1937 Fisher, and independently Kolmogorov, Petrovsky and Piskunov, studied the travelling wave of an advantageous gene, u_t = Du_xx + ru(1 − u) [26, 27]. Waves exist only for speeds $c \geq 2\sqrt{rD}$. A simulation confirms the minimal speed. With r = D = 1, explicit finite differences (dx = 0.1 on [0, 400], dt = 0.004, step initial data u = 1 for x below 10) move the u = 0.5 level at 1.981 between t = 100 and t = 150, against the predicted 2.000. The two works were parallel.

In 1945 Leslie introduced the Leslie matrix [28, 29, 30]. The growth rate of an age-structured population is its dominant eigenvalue, and the stable age distribution is the corresponding eigenvector. Neither Bacaër nor Wikipedia names the Perron–Frobenius theorem here. The matrix puts Lotka’s stable-population theory into discrete time, according to Bacaër [28]. Wikipedia gives its characteristic polynomial as the Euler–Lotka equation [29]. Leslie’s own abstract starts from “a stable Malthusian distribution” but does not name Lotka [30].

Patterns, nerves, neutral evolution and genealogies

In 1952 Turing showed how two reacting and diffusing morphogens form patterns (*Phil. Trans. R. Soc. B* 237, 37–72, 14 August 1952) [31]. A state that is stable without diffusion becomes unstable when the diffusion rates differ, and stationary patterns appear. In Turing’s words, a homogeneous system “may later develop a pattern or structure due to an instability of the homogeneous equilibrium”. The paper rests on no earlier link. Kondo and Asai’s 1995 study of the angelfish pattern is experimental support, not a proof [32, 33]. This is the view of Kondo’s review, and independently of Crampin, Hackborn and Maini (2002), who model the pattern as a reaction–diffusion scheme.

In 1952 Hodgkin and Huxley wrote four nonlinear differential equations with gating variables m, h and n [34, 35, 36]. Fitted to voltage-clamp data on the squid giant axon, they reproduce the action potential. The claim that they also matched the conduction velocity in 1952 is not supported (note 10). The model rests on no earlier link.

FitzHugh (1961) and Nagumo and colleagues (1962) reduced this to a two-variable model of excitability [37, 34, 38]. FitzHugh’s original paper (*Biophys. J.* 1, 445–466) uses the BVP model $dx/dt = c(y + x - x^3/3 + z)$, $dy/dt = -(x - a + by)/c$. He pairs (V, m) as the fast variables and replaces n and −h by w = 0.5(n − h). He calls the (u, w) plane a projection, “not a phase plane at all” [37]. The textbook form $\tau\, du/dt = u - u^3/3 - w + RI$, $\tau_w\, dw/dt = b_0 + b_1 u - w$ treats m as instantaneous [34]. The model rests on Hodgkin–Huxley 1952, by reduction.

In 1968 Kimura proposed the neutral theory: most substitutions are fixed by drift [39, 10]. King and Jukes proposed it independently in 1969, in “Non-Darwinian evolution” in *Science* [40, 39]. The fixation probability shows the difference selection makes. For a single new mutant, p = 1/(2N), with N = 100 and s = 0.01, the probability is (1 − e^(−2s))/(1 − e^(−4Ns)) = 0.02017, against 0.005 for a neutral mutant. Kimura’s

1962 paper (*Genetics* 47, 713–719) sets p = 1/(2N) and prints the approximation 2s/(1 − e^(−4Ns)) = 0.02037 [41] (note 11). The theory builds on the framework of Fisher, Haldane and Wright.

In 1973 Maynard Smith and Price defined evolutionarily stable strategies, applying game theory to fights within a species [42, 43]. The standard example is the Hawk–Dove game. The paper is "The logic of animal conflict", *Nature* 246 (1973), 15–18 [42]. The SEP notes that Maynard Smith's 1972 chapter came first, and that the 1973 paper put the idea into wide circulation [43]. It rests on no earlier link in this chain.

In 1974 May showed chaos in the discrete logistic map $x \to rx(1 - x)$ [44, 45]. As r grows, a sequence of bifurcations leads to chaos, so long-term prediction becomes impossible. At r = 4 the Lyapunov exponent, averaged over 10^6 iterations from x_0 = 0.1234, is 0.6931 = ln 2. At r = 3.2 the map settles on the 2-cycle 0.7995, 0.5130, matching the closed form $((r+1) \pm \sqrt{(r+1)(r-3)})/(2r)$. The map is Verhulst's logistic in discrete time: Wikipedia says May "introduced … the logistic map, by discretizing the logistic equation in time" [44, 45].

In 1981 Felsenstein gave maximum-likelihood phylogenies (*J. Mol. Evol.* 17, 368–376, November 1981) [46, 47, 48]. The likelihood of a tree is computed by dynamic programming over subtrees, a method called pruning. Exhaustive search is hopeless because the number of unrooted binary trees with n leaves is (2n − 5)!!: for n = 4 to 10 this gives 3, 15, 105, 945, 10395, 135135 and 2027025. The paper builds on Felsenstein's own 1973 likelihood algorithm, and cites Edwards and Cavalli-Sforza (1964) for the number of unrooted trees (note 12).

In 1982 Kingman introduced the coalescent [49, 10, 50]. It describes the genealogy of a sample under the Wright–Fisher model as the effective population size Ne grows, with a pair of lineages coalescing at rate 1/(2Ne) per generation. For n = 10 and Ne = 1000 the expected time to the most recent common ancestor is 3600 generations, and for a pair 2000. The coalescent rests on the Wright–Fisher model of 1931, taken as a large-population limit with time measured in units of N generations [49, 50].

In 2025–2026 Girardin and Nadin proved the existence of planar pulsating travelling waves for non-cooperative Fisher–KPP systems in media that are periodic in space and time [51]. The paper is a preprint as of 29 October 2026 (arXiv 2506.22003, submitted 27 June 2025, v2 on 6 February 2026, with no journal reference or DOI). It rests on Fisher–KPP 1937 (note 13).

Notes

1. Only Bacaër's book discusses both Fibonacci and Malthus, and it draws no line between them [1, 4].
2. MacTutor says Verhulst argued against Quetelet's model, while Wikipedia says he worked under Quetelet's guidance; this is not settled from a primary source [5, 7]. The two agree in substance on the naming of the logistic, cited here as 1844 (published 1845).
3. MacTutor names only Pearl and Reed (1920) as rediscoverers; Wikipedia names McKendrick (1911) first [5, 7]. Both are kept.
4. Wikipedia dates Pearson's equilibrium with p = q = 0.5 to 1903; the common citation is 1904. This is unsettled.
5. Only the SEP states that Fisher 1918 rests on Hardy–Weinberg [10]. Visscher and Goddard set up Fisher's single-locus model with genotypes in Hardy–Weinberg

equilibrium, but do not attribute the assumption to Fisher in so many words [14]. MacTutor confirms the paper, not this role [13].

6. Bacaër places Bienaymé (1845–1847) before Galton and Watson (1873–1875), so the branching process that Fisher and Haldane use has an earlier, independent root [20].
7. That Haldane took the branching process from Fisher 1922 is not stated by any source. Charlesworth and Krukov–de Koning confirm each method, and Patwa and Wahl group Fisher 1922 and Haldane 1927 under one model, but none states the dependency [19, 21, 22].
8. Haldane's selection series ran from 1924 to 1934, with Part V ("Selection and mutation") in 1927 [52, 53]. The SEP's dates of 1930–1932 cover only the later parts.
9. The dependence of Lotka–Volterra on Verhulst's logistic is stated in one source for Lotka and in another for Volterra; it is not otherwise confirmed. Wikipedia's claim that Lotka used Kolmogorov's framework in 1920 is impossible, since Kolmogorov was born in 1903, and it is not used [15].
10. Huxley's 2002 retrospective reports only earlier runs with conduction velocities that were too high, so no source supports a 1952 match of conduction velocity [36].
11. Only p = 1/(2N) and the small-s form are legible in the scan of Kimura 1962 used here; Kimura's general fixation formula was not checked against it [41].
12. Felsenstein's debt to his 1973 algorithm and to Edwards and Cavalli-Sforza 1964 is stated only in his own paper [47]. The paper does not cite Cavalli-Sforza and Edwards 1967.
13. The link from Girardin–Nadin to Fisher–KPP is stated only in the paper itself [51]. No 2025–2026 milestone with two independent sources was found. Fisher's fundamental theorem (1930) is left out of this chapter.

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Control theory

Control theory asks how to steer a system with feedback so that it stays stable and does what is wanted. The chain starts with governors: Ktesibios' water clock, then Huygens and Watt. It continues with Maxwell's request in 1868 for stability conditions above third order, answered by Routh (1877) and Hurwitz (1895), alongside Lyapunov's second method (1892).

The main lines that follow are four. Frequency methods for feedback amplifiers came from Black, Nyquist and Bode. Filtering and state space ran from Wiener to Kalman (1960) and Kalman–Bucy (1961). Optimal control came from Bellman and Pontryagin. After Doyle showed in 1978 that LQG controllers have no guaranteed margins, robust control (Zames, Kharitonov, Doyle–Glover–Khargonekar–Francis) and model predictive control took over.

In September 2026 two independent preprints proved the 1997 conjecture of Blondel and Tsitsiklis that static output-feedback stabilization is NP-hard; both are still preprints as of 28 October 2026. There is no arrow between Bellman and Pontryagin, between Strecker and Nyquist, or between the two 2026 proofs. Black's amplifier influenced Nyquist's work, but no source says it led to the criterion itself.

Governors and stability: Ktesibios to Lyapunov

Ktesibios, who flourished between 285 and 222 BCE, built a feedback mechanism into water clocks [1, 2, 3]. From him to Maxwell is 2152 years, counting from 285 BCE and allowing for the absence of a year 0.

The centrifugal governor came next, first for windmills and then for steam engines [4, 5, 6]. It is a form of proportional control. It is credited to Huygens in the 17th century, and Watt adapted it in 1788; he did not invent it (note 1).

In 1868 Maxwell published "On Governors" (Proc. Roy. Soc. 16) [5, 4]. He showed that a linearised system is stable when all roots of its characteristic equation have negative real part. He gave conditions up to third order and asked mathematicians for higher orders. For a cubic $s^3 + a_1 s^2 + a_2 s + a_3$ the condition is $a_1 a_2 > a_3$. For example, $s^3 + 2s^2 + 3s + 5$ has $a_1 a_2 = 6$, above 5, and its roots -1.844 and $-0.078 \pm 1.645i$ are indeed all stable. The governor posed the problem Maxwell solved.

In 1877 Routh gave his test in his Adams Prize treatise [7, 8] (note 2). The number of roots in the right half-plane equals the number of sign changes in the first column of the Routh array. A random test bears this out. For 2000 random monic polynomials of degree 2 to 6, with coefficients uniform on $[-1, 3]$ (numpy default_rng, seed 0; cases with a zero pivot or a root within 10^{-9} of the axis skipped), the sign changes equal the count of right-half-plane roots in all 2000 cases. The test rests on Maxwell 1868, which posed the problem, and generalises his conditions to all linear systems.

In 1892 Lyapunov wrote *The general problem of the stability of motion*, the thesis he defended on 12 October 1892 [9, 10]. Its second method shows that an energy-like function V that does not increase along trajectories implies stability. It rests on no earlier link in this chain.

In 1895 Hurwitz proved that a polynomial is stable if and only if all leading Hurwitz determinants are positive (Math. Ann. 46) [11, 12, 6]. He was answering a question of Stodola, who had published the criterion without proof in 1894 [12, 6]. In the same random run as for Routh, positive Hurwitz minors and stability coincide in 2000 of 2000 cases, so the two criteria agree. Whether Hurwitz generalised Routh or worked independently is disputed (note 3).

Feedback amplifiers: Black, Nyquist and Bode

In 1927 Harold Black conceived the negative-feedback amplifier, which trades gain for linearity [13, 14, 15]. He had the idea on 2 August 1927 (note 4). His first application was filed on 8 August 1928, and Kline's reference list gives 22 April 1932 as the filing date of the issued patent. The patent, US 2,102,671, was issued on 21 December 1937, and the paper appeared in BSTJ 13 (1934). From filing to issue is 9.37 years from the 1928 application and 5.66 years from the 1932 one.

In 1932 Nyquist published "Regeneration Theory" in BSTJ [16, 17, 15]. The loop is stable if and only if the plot of $AJ(i\omega)$ does not enclose the point $1 + i0$. This is his sign convention; the modern form uses -1. He derived it by contour integration and residues, and Wikipedia's account via the argument principle is compatible. A numerical test works as the criterion predicts. For $L = K/(s + 1)^3$, counting the winding of $1 + L(i\omega)$ about 0 on a grid of 4,000,001 frequencies from -1000 to 1000, $K = 4$ and $K = 7.9$ give 0 encirclements and 0 closed-loop poles in the right half-plane; $K = 8.1$ and $K = 12$ give 2 and 2. The border lies at $K = 8$.

The criterion rests on residues and the argument principle of complex analysis. It was also influenced by Black's work. According to Kline, Nyquist analysed Black's feedback design after a meeting in May 1928, and this led to his criterion [15]. Guarnieri, independently, calls Black's negative feedback "seminal to subsequent work done at AT&T by Harold Nyquist", who developed the criterion in 1932 [18]. According to Wikipedia, Strecker found the same criterion independently in 1930, so there is no arrow between them (note 5).

Between 1938 and 1945 Bode developed the Bode plot (1938), the gain–phase relations (1940), gain and phase margins, and his 1945 book on feedback amplifier design [19, 17, 20, 15]. All of this comes from constraints that analytic functions place between gain and phase. One such constraint is Bode's sensitivity integral. For $L = 4/(s + 1)^3$ the integral of $\ln |S(j\omega)|$ over $[0, \infty)$ is 0 to within 10^{-13}, as the theorem requires. The work rests on Nyquist 1932: Bode's 1940 paper applies "the Nyquist stability diagram", citing BSTJ of July 1932 [20].

Filtering, state space and optimal control

Wiener's filter dates from a report of 1942 and a book of 1949 [21, 22, 23, 24] (note 6). It solves the stationary Wiener–Hopf equation by spectral factorisation. In the scalar case with $a = -1$, $q = 2$ and $r = 1$, the filter pole is $-\sqrt{(a^2 + q/r)} = -1.732051$. It rests on no earlier link here.

Bellman developed dynamic programming, with a first paper in 1952 and a RAND book in 1953 [25, 26, 27] (note 7). The principle of optimality is stated in his 1957 book [26]. The Bellman equation for stochastic control followed in 1958 [25]. It rests on no earlier link in this chain.

In 1956 Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko found the maximum principle [28, 29, 30, 60]. Along an optimal trajectory the Hamiltonian is maximised, and the classical conditions of the calculus of variations follow. The book appeared in 1961, and the authors received the Lenin Prize in 1962. The principle generalises the calculus of variations [28, 29]. According to Wikipedia, it is related to Bellman’s dynamic programming but was developed in parallel, so there is no arrow between them (note 8).

In 1960 Kalman introduced controllability and observability, with rank conditions in state space [31, 22, 32, 33]. The paper was “On the General Theory of Control Systems”, at the 1st IFAC World Congress. The rank conditions are easy to test. The 3×3 companion system of $s^3 + 3s^2 + 2s + 1$ with $B = e_3$ has controllability rank 3; diag(1, 2) with $B = (1, 0)^{\mathrm{T}}$ has rank 1. The paper rests on no earlier link here.

Also in 1960 Kalman published the Kalman filter (Trans. ASME J. Basic Eng. 82) [21, 22, 34, 35, 36]. It uses orthogonal projection on a state-transition model, with a Riccati difference equation in place of the Wiener–Hopf equation. It is dual to the noise-free regulator problem. For a scalar random walk with $q = 0.1$ and $r = 1$, the iterated Riccati prior converges to 0.370156, equal to the closed form. The filter rests on Wiener 1942/1949, whose limits it removes; the paper cites Wiener [21, 22]. It also rests on the noise-free regulator problem, by the duality the paper states. Swerling (1958), Thiele (1880) and Stratonovich (1959–60) did similar work, but independently, so there is no arrow from them [34, 35, 36] (note 9).

In 1961 Kalman and Bucy gave the continuous-time filter [37, 38]. It uses a Riccati differential equation whose equilibrium gives Wiener’s stationary solution. With $a = -1$, $q = 2$ and $r = 1$ the equilibrium variance is $P = 0.732051$ and the filter pole is -1.732051, the same as Wiener’s pole. The filter rests on Kalman 1960, carried from discrete to continuous time; the paper cites it, and Wikipedia calls Kalman–Bucy the continuous-time form. It also contains Wiener’s filter as a special case, the equilibrium of the variance equation (note 10).

In 1967 Wonham proved that poles can be placed arbitrarily if and only if the pair (A, B) is controllable, for real systems with several inputs [43, 44]. The single-input case goes through the companion form. Two examples confirm this. Take A with ones on the superdiagonal and last row (1, 2, 3, 4), and $B = (e_2, e_4)$: the controllability rank is 4, and scipy’s place_poles places the poles exactly at -1, -2, -3 and -4. For the single-input companion form of $s^3 + 3s^2 + 2s + 1$, Ackermann’s gain $K = (23, 24, 6)$ gives poles -2, -3 and -4. The proof uses controllability as the rank of [B, AB, …], the notion Kalman introduced, although the report cites Gantmacher and Lefschetz for it and names Kalman only on design criteria [43] (note 11). For sufficiency it follows Langenhop’s construction, which the report credits [43].

Model predictive and robust control

Model predictive control grew in steps from Propoi in 1963 to the review by Mayne and colleagues in 2000 [39, 40, 41, 42]. The steps were the moving horizon (1963), IDCOM (1976/78), DMC (1979/80), QDMC (1983/86) and a stability theory in the 1990s. The controller repeatedly solves a finite-horizon problem, and Lyapunov arguments give stability. A small simulation shows the idea. A double integrator sampled at 0.2 ($A = [[1, 0.2], [0, 1]]$, $B = (0.02, 0.2)^{\mathrm{T}}$), with input bounded by 1, horizon $N = 10$, $Q = I$, $R = 1$ and the LQR terminal cost, each step solved by L-BFGS-B, goes from (5, 0) to

$(-1.4\times10^{-4}, 4.2\times10^{-4})$ in 60 steps, and the optimal value never increases.

Model predictive control rests on three things. It was set against LQG, which ignored constraints, according to Qin and Badgwell and to Rawlings [39, 40]. It rests on Lyapunov 1892, since the optimal value function serves as a Lyapunov function, as Grüne's notes and the NTNU sources show [42, 39, 40]. And it rests on the steps from Propoi to QDMC, described by Qin and Badgwell; Lee (2011) cites Propoi 1963, DMC 1980 and QDMC [39, 41].

In 1975 Anderson, Bose and Jury showed that stabilizability by output feedback is decidable, by recasting it so that known decision algorithms apply [45, 46]. A small case shows why output feedback is harder than state feedback. The double integrator with position output has no stabilising static gain: over gains k from -100 to 100 in steps of 0.01, the best largest real part of the eigenvalues is 0.0. State feedback with $K = (2, 3)$ gives poles -1 and -2. The result rests on no earlier link in this chain.

From 1976 to 1981 Zames founded H∞ control [47, 48, 49]. He posed feedback as an optimisation in an operator norm, the problem of sensitivity minimisation (IEEE TAC 1981). Helton (1979) and Tannenbaum (1977) did related work at the same time [47]. The standard H∞ problem reduces to the Nehari problem of approximating a bounded function by one in H∞, so it rests on that problem [47, 49].

In 1978 Doyle showed that LQG regulators have no guaranteed margins [50, 51]. His counterexample is a two-state system with one input and one output, whose gain margin goes to 0. A scan reproduces it. With $A = [[1, 1], [0, 1]]$, $B = (0, 1)^{\mathrm{T}}$, $C = (1, 0)$, $Q = q(1, 1)^{\mathrm{T}}(1, 1)$, process-noise intensity $\sigma(1, 1)^{\mathrm{T}}(1, 1)$ and $R = V = 1$, the input gain m is scanned from 0.5 to 1.5 in steps of 5×10^{-6}. The stable range is [0.921, 1.056] at $q = \sigma = 1$ and shrinks to [0.973, 1.0009] at $q = \sigma = 1000$, while LQR alone stays stable for gains from 0.5 to 100. The counterexample rests on the LQG loop, a Kalman filter plus LQR, and on the LQR margins of Safonov and Athans (1977), which it shows do not carry over.

In 1978 Kharitonov proved that an interval polynomial is Hurwitz if and only if four vertex polynomials are [52, 53]. For $s^3 + [3, 4]s^2 + [5, 6]s + [2, 3]$, all four Kharitonov polynomials are stable, and so are 5000 random members (coefficients uniform in their intervals, numpy default_rng, seed 0). The four polynomials are tested with Routh–Hurwitz, so the theorem uses that criterion (note 12).

In 1987–88 Bartlett, Hollot and Huang Lin proved the edge theorem: for a polytope of polynomials it suffices to check the exposed edges [53, 54, 55] (note 13). It was presented at the American Control Conference of 1987 and published in *Mathematics of Control, Signals and Systems* 1 (1988) 61–71. The Encyclopedia of Mathematics says the edge theorem covers a much more general class than Kharitonov's theorem [53]. Fu and Barmish note that the edge criterion "does not easily degenerate into Kharitonov's Theorem" [54]. So it extends Kharitonov's line of work rather than depending on his proof.

In 1989 Doyle, Glover, Khargonekar and Francis gave state-space solutions to the standard H2 and H∞ problems [56, 48]. The solution needs two Riccati equations, a spectral-radius condition and a parametrisation by linear fractional transformations. It solves the H∞ problem Zames had posed, so it builds on Zames 1981, although no source states the dependency (note 14). Doyle's 1978 result is often seen as the motivation, but no source states that either, so no arrow is drawn.

Complexity: Blondel–Tsitsiklis and the 2026 proofs

In 1997 Blondel and Tsitsiklis proved that static output feedback with gains constrained to intervals is NP-hard (SICON 35) [46, 57]. They left the unconstrained case open and conjectured that it is NP-hard too. The paper cites Anderson, Bose and Jury 1975 as its reference [1]: the problems are decidable by Tarski–Seidenberg elimination, though not in polynomial time (note 15).

In 2026 Löfberg proved that unrestricted static output-feedback stabilization is NP-hard, even with a single input and several outputs [58]. The proof is a polynomial reduction from the Betweenness problem. It appeared on arXiv on 15 September 2026 and is a preprint as of 28 October 2026. It answers the open question of Blondel and Tsitsiklis; the preprint says that "to the best of our knowledge" it resolves it (note 16).

Two days later, on 17 September 2026 (v2 on 20 September), Ahmadi, Chaudhry, Narang and Tang posted their proof [57, 59]. No polynomial or pseudo-polynomial algorithm decides stabilization by a linear output-feedback controller unless P = NP, in continuous and in discrete time. The proof goes through 3SAT and simultaneous stabilization. The paper's AI Declaration says that ChatGPT 5.6 produced a one-shot NP-hardness proof, which the authors realised on 2 August 2026. The paper appeared on Optimization Online on 30 September, updated on 26 October, and is a preprint as of 28 October 2026. It comes 29 years after Blondel and Tsitsiklis, whose paper it cites as [2] and whose conjecture it proves. It does not cite Löfberg, so the two 2026 proofs were independent.

Notes

1. The governor is credited to Huygens, with Watt only adapting it, by Wikipedia and by Chen (CityU) [4, 5]. The Lund lecture instead credits windmill governors to Lee (1745) and Mead (1787), and gives Watt as 1788 and 1798 [6]. This is not settled.
2. Routh's test is dated 1874 and 1876 by two Wikipedia pages, and 1877 by MacTutor and the Encyclopedia of Mathematics, which cite the treatise [7, 8]. The Lund lecture places it in the Adams Prize competition of 1875–77, with the treatise dated 1876 and Maxwell as judge [6]. This is not settled from a primary source. Hurwitz is dated 1895 from the Encyclopedia of Mathematics citation; Wikipedia's 1877 conflates him with Routh.
3. The Encyclopedia of Mathematics calls Hurwitz's criterion a generalisation of Routh's, while Wikipedia says Hurwitz proposed it independently [11]. Both versions stand; the criteria agree numerically.
4. Black's idea is dated 2 August 1927 by ETHW and Kline (IEEE Control Systems, 1993), and 6 August by Wikipedia; 2 August is kept [13, 14, 15]. The patent was filed on 8 August 1928 and issued on 21 December 1937 (Wikipedia, Kline); Kline's reference list gives a 1932 filing date for the issued patent number.
5. Only Wikipedia states Strecker's independent discovery of 1930 [17].
6. No source consulted gives the month of Wiener's 1942 report [21, 23, 24].
7. Wikipedia dates Bellman's dynamic programming to the 1940s, against first publication in 1952, the RAND book of 1953 and the book of 1957 [25, 26].
8. Only Wikipedia states that the maximum principle and dynamic programming developed in parallel [29]. Gamkrelidze's own history of the discovery describes

it as the work of Pontryagin's department in the mid-1950s [60].

9. The 1960 filter paper does not mention controllability or observability; its duality is with the regulator problem, so the state-space concepts of Kalman's IFAC paper are not an input to the filter [21].
10. That Wiener's filter is the equilibrium case of Kalman–Bucy is stated in §2 and Theorem 5 of the paper itself [37].
11. Wonham's report cites Lefschetz, Langenhop, Gantmacher, Birkhoff–MacLane and Kalman 1964 ("When is a linear control system optimal?"), the last only for design criteria [43]. The link to Kalman's notion of controllability rests on the content, not on a citation.
12. The Encyclopedia of Mathematics gives the theorem, and Wikipedia the use of Routh–Hurwitz to test the four polynomials [52, 53].
13. The third author appears as "H. Lin" in Fu and Barmish (1988) and as "L. Huang" in later citations; the name is Huang Lin [54, 55].
14. The abstract of the 1989 paper names neither Zames nor Doyle 1978 [56].
15. The link from Blondel–Tsitsiklis to Anderson–Bose–Jury is stated in the 1997 paper itself [46].
16. Only the arXiv preprint itself supports Löfberg's result; Ahmadi and colleagues do not cite it [58, 57]. Neither 2026 proof has been peer-reviewed. Not researched for this chapter: Popov, the circle criterion, the KYP lemma, the small-gain theorem, Brockett and Coron, and Kalman–Bertram; Keerthi–Gilbert and Mayne et al. were seen only through secondary sources. The MSC classes covered are 93A General systems theory, 93B Controllability, observability and system structure, 93C Model systems in control theory, 93D Stability of control systems and 93E Stochastic systems and control (MathSciNet; an older list on the Belgrade mirror words 93A, 93C and 93D differently).

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58. arXiv 2609.16886, Löfberg, NP-hardness of static output-feedback stabilization
59. Optimization Online, Complexity of output feedback stabilization
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Information theory

Information theory measures information and sets the limits of how well it can be compressed, kept secret and sent through noise. The chain starts with al-Kindi's frequency analysis (c. 850) and with Nyquist and Hartley (1924–1928). Shannon took over their logarithmic measure in his 1948 paper, which is the hub of the field.

From Shannon three lines run. The first is secrecy (Shannon 1945/1949). The second is source coding (Kraft, Huffman, McMillan). The third is channel coding, from Golay and Hamming through Reed–Muller, BCH, Reed–Solomon and Gallager's LDPC codes to turbo codes, polar codes and the proofs (2016–2023) that Reed–Muller codes reach capacity. A combinatorial side line runs from Shannon's zero-error capacity of 1956 to Lovász's $\Theta(C_5) = \sqrt{5}$ (1979).

In 2026 a burst of preprints raised the lower bound on $\Theta(C_7)$ to 3.25883262 (31 August 2026), but $\Theta(C_7)$ itself is still open as of 28 October 2026. There is no arrow between BCH and Reed–Solomon codes, which were found independently. MacKay and Neal rediscovered Gallager's codes rather than building on them.

Foundations: al-Kindi to Shannon 1948

Around 850 al-Kindi described frequency analysis, in the earliest known treatise on cryptanalysis [1, 2, 3, 4]. The method matches the letter frequencies of a ciphertext to those of the language. A small test shows it works. Shift the opening sentence of Shannon's 1948 paper ("The recent development of various methods of modulation …") by 7 places. The most frequent cipher letter is l, 18 times, and mapping it to e recovers the whole text. The treatise was rediscovered in 1987 in the Sulaimaniyyah Ottoman Archive in Istanbul [4, 3].

Nyquist's papers of 1924 and 1928 ("Certain Topics in Telegraph Transmission Theory") set the telegraph pulse limit at twice the bandwidth [5, 6]. In 1928 Hartley's "Transmission of Information" (BSTJ) argued that information grows with the logarithm of the number of distinguishable levels [5, 6]. Hartley's measure ignores probabilities. For probabilities (½, ¼, ⅛, ⅛), Shannon's entropy is 1.75 bits, against Hartley's $\log_2 4 = 2$. Neither rests on an earlier link.

In 1937/38 Shannon's MIT thesis, "A Symbolic Analysis of Relay and Switching Circuits" (Trans. AIEE), showed that relay networks correspond to Boolean expressions [7, 8, 9]. It rests on Boole's algebra. Nakashima's approach in Japan was parallel and two years earlier, and was developed at first without knowledge of Boole, so there is no arrow between them [9].

In 1948 Shannon published "A Mathematical Theory of Communication" (BSTJ 27) [5, 6, 17, 18]. He showed that natural axioms force the entropy to be $H = -K \Sigma\, p \log p$. He proved the noisy-channel coding theorem with capacity C by random coding. Some capacities make this concrete. A binary symmetric channel with error rate p has capacity $1 - h(p)$: 0.919207 at p = 0.01, 0.500084 at p = 0.11 and 0 at p = 0.5. A Gaussian channel at signal-to-noise ratio 15 carries $\tfrac{1}{2} \log_2(1 + 15) = 2$ bits per real sample, or 4 bits per complex sample. The paper rests on Nyquist and Hartley for the

logarithmic measure. It also uses Hamming's code as an example, "found by a method due to R. Hamming", before Hamming's own paper of 1950 [5, 17]. Fano found his coding method independently, so there is no arrow from Fano (note 1).

Secrecy and source coding

Shannon's "Communication Theory of Secrecy Systems" was written as a classified memorandum dated 1 September 1945 and published in 1949 (BSTJ 28(4), 656–715) [10, 11, 12, 13, 14] (note 2). It defines perfect secrecy: the posterior probability of each message equals its prior. It proves that perfect secrecy needs at least as many keys as messages, and that the Vernam system is perfect. For a one-time pad on 2-bit messages, each of the 16 message–ciphertext pairs occurs exactly once, so P(C = c given M = m) is uniform. The paper rests on Vernam's system of 1926, which it cites in §4.3 and footnote 6 [10, 15]. It also uses the entropy of the 1948 paper in its §9, for example in the unicity distance [10, 16]; the 1945 report predates the 1948 publication.

In 1949 Kraft proved the Kraft inequality Σ 2^(−lᵢ) ≤ 1 for prefix codes, in his MIT master's thesis [19, 20, 21, 22]. According to Wikipedia, Kraft attributes the analysis to Raymond Redheffer (note 3). The Huffman lengths (2, 2, 2, 3, 3) meet it with equality: Σ 2^(−l) = 1.0. The inequality rests on no earlier link.

In 1952 Huffman published minimum-redundancy codes (Proc. IRE) [24, 25]. The code is built from the bottom up by repeatedly merging the two least probable symbols. It began as a term paper in Fano's course of 1951. For p = (0.4, 0.2, 0.2, 0.1, 0.1) the Huffman average length is 2.2, equal to the optimum found by brute force over all lengths that satisfy Kraft's inequality; the entropy is 2.1219. The work rests on Fano, who posed the problem; his own method was not always optimal.

In 1956 McMillan proved that the Kraft inequality holds for all uniquely decodable codes, not only prefix codes, in "Two inequalities implied by unique decipherability" [19, 21, 30]. The statement generalises Kraft's, as Foldes notes [21]. But McMillan found it independently, according to Wikipedia, so his proof does not use Kraft's and there is no arrow [19].

Algebraic codes: Golay to Gallager

In 1949 Golay found the perfect triple-error-correcting (23, 12, 7) code (Proc. IRE 37) [17, 23]. The code is perfect because 2^{12} times the sum of C(23, i) for i ≤ 3 is 8,388,608 = 2^{23}. The cyclic code with generator g = $x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2 + 1$ has minimum weight 7 over all 4095 nonzero codewords. Costello and Forney place Golay's work shortly after Shannon's paper, which suggests an influence (note 4).

In 1950 Hamming published single-error-correcting codes built from parity checks (BSTJ 29) [23, 17]. The checks spell out the position of an error. He also gave the condition 2^k ≥ m + k + 1, the sphere bound and the Hamming distance. The Hamming (7, 4) code corrects all 112 single errors in its 16 codewords and has minimum distance 3. Since $2^3 = 8 \geq 4 + 3 + 1$ and $16 \cdot 8 = 2^7$, it is perfect. Hamming acknowledged Golay 1949 as prior work, but his proof does not depend on it; the link is one of order only (note 5).

In 1954 Muller and Reed gave the Reed–Muller codes, the evaluations of low-degree

Boolean polynomials, with Reed's decoding scheme (Trans. IRE PGIT 4) [26, 27]. Reed and Solomon later noted that their codes reduce to these over Z_2 [26]. The first-order codes RM(1, m) for m = 3, 4 and 5 have distance 4, 8 and 16, which is 2^(m−1), as an exhaustive check over all codewords confirms. They rest on no earlier link here.

In 1959 Hocquenghem, and independently in 1960 Bose and Ray-Chaudhuri, found BCH codes [31, 17]. These are cyclic codes with prescribed roots and a designed distance. For BCH(15, 7) the generator $g = m_1 m_3$ has degree 8 and minimum weight 5, equal to the designed distance. The two discoveries were independent, so there is no arrow between them.

In 1960 Reed and Solomon published polynomial codes over finite fields (J. SIAM 8 (1960) 300–304) [26, 32, 17]. The message polynomial is evaluated at all field elements, and decoding is by plurality. The codes are MDS: they meet the Singleton bound. For example, RS[7, 3] over GF(8) has minimum weight $5 = n - k + 1$. The paper cites Hamming 1950 and Reed 1954 as its references [1] and [2] (note 6). BCH codes were found independently, so there is no arrow from them.

Gallager's low-density parity-check codes date from his 1960 thesis, a 1962 paper and a 1963 monograph, three stages of one work [17, 33]. The codes use sparse parity-check matrices and iterative (APP) decoding by message passing. They rest on Shannon 1948, which posed the problem of reaching capacity.

Zero-error capacity: Shannon 1956 to the 2026 bounds

In 1956 Shannon defined the zero-error capacity of a channel, using strong products of its confusion graph [28, 29]. For the pentagon he showed $\tfrac{1}{2} \log 5 \le C_0 \le \log(5/2)$, with C_0 itself unknown. The lower bound comes from the set {(i, 2i mod 5)}, which is independent in $C_5 \boxtimes C_5$. An exhaustive search confirms $\alpha(C_5^2) = 5$, and the bounds are $\log_2 \sqrt{5} = 1.160964 \le C_0 \le \log_2(5/2) = 1.321928$. The paper rests on no earlier link here.

In 1979 Lovász solved the pentagon in "On the Shannon capacity of a graph" (IEEE Trans. IT 25) [34, 35]. Orthonormal representations give a number θ with $\theta \ge \Theta$, and $\theta(C_5) = \sqrt{5}$, so $\Theta(C_5) = \sqrt{5}$. The formula $\theta(C_5) = 5 \cos(\pi/5)/(1 + \cos(\pi/5))$ gives $\sqrt{5} = 2.236068$ exactly. According to a 2023 survey, this is still the only known capacity of a graph that is not perfect [35]. It rests on Shannon 1956, whose open problem it solved.

In 2019 Polak and Schrijver proved $\Theta(C_7) \ge 367^{(1/5)}$, from an explicit independent set of 367 points in C_7^5 [45, 46]. The value is $367^{(1/5)} = 3.257865967$. For comparison, a branch-and-bound search gives $\alpha(C_7^2) = 10$, so the square alone only gives $\sqrt{10} = 3.162278$. The work rests on Lovász 1979, which framed the open problem and supplies the upper bound.

In 2026 Itty, Rosin, Carstensen and Reichman proved $\Theta(C_7) \ge 134753^{(1/10)}$, above 3.258020 [50, 46]. They also gave new bounds for C_{11}, C_{13} and C_{15}. The constructions were found through iterative interaction with a large language model. The paper appeared on arXiv on 23 July 2026 and is a preprint as of 28 October 2026. The stated values all hold: $134753^{(1/10)} = 3.258020737$, $21909^{(1/6)} = 5.289774$, $62530^{(1/6)} = 6.300109$ and $8076974^{(1/8)} = 7.301399$. The work improves the record of Polak and Schrijver (3.258021 > 3.257866); Gao's abstract names both constructions [46]. It rests on Lovász 1979 for the upper bound.

A week later Gao proved $\Theta(C_7) \geq 3.2587891539\ldots$ (arXiv, 30 July 2026, a preprint as of 28 October 2026) [46, 51, 52]. A product lemma combines gadgets into an independent set in C_7^200. The proof rests on Polak and Schrijver 2019, whose set in C_7^5 is the base gadget, and on Itty et al. 2026, which it reformulates and extends, according to Gao's abstract and Tandon's.

On 31 July 2026 Buys, Polak and Zuiddam gave lower bounds formalised in Lean: $\Theta(C_7) \geq 3.258805369885\ldots$, with bounds for six more odd cycles [51, 52]. The paper is a preprint as of 28 October 2026. It rests on Gao 2026, itself based on Itty et al., according to its abstract; Tandon's abstract describes the Buys–Polak–Zuiddam framework as built on Gao's.

On 31 August 2026 Tandon raised the bound to $\Theta(C_7) \geq 3.25883262\ldots$ [52]. He refines Gao's and the Buys–Polak–Zuiddam recursive constructions so that different parts of the recursion can use different independent sets. This gives an independent set in C_7^⊠500. The paper is a preprint as of 28 October 2026. The gap to Lovász's upper bound $\theta(C_7) = 3.317667207$ is still 0.058835. The work rests on Itty et al., on Gao 2026 and on the Buys–Polak–Zuiddam framework (note 7).

Capacity-approaching codes: turbo, LDPC, polar and Reed–Muller

In 1993 Berrou, Glavieux and Thitimajshima introduced turbo codes [17, 36, 37]. Parallel concatenation plus iterative decoding comes close to the Shannon limit. The results were first met with disbelief, then confirmed by other laboratories. The patent, US 5,446,747, names Berrou alone; it claims French priority of 23 April 1991, was filed in the US on 16 April 1992, granted on 29 August 1995 and expired on 29 August 2012, some 21 years after the priority date [37] (note 8). Turbo codes rest on Battail's probabilistic processing, which Berrou and Costello–Forney both credit [17, 36]. Berrou names Gallager 1962 and Tanner 1981 as precursors with closely related principles; Costello and Forney link Tanner graphs to turbo decoding only later, through Wiberg 1995 [17, 36].

In 1996 MacKay and Neal rediscovered LDPC codes [17, 33, 38]. They showed empirically that belief propagation on sparse graphs comes close to the Shannon limit, 34 years after Gallager's 1962 paper. The trigger was the turbo codes of 1993. Gallager's work was reinvented and then credited ("reinvented Gallager's decoding algorithm and GL codes"), not built on, so there is no arrow from Gallager.

In 2008/2009 Arıkan introduced channel polarization [39, 40, 41]. Polar codes reach the symmetric capacity of any binary-input memoryless channel, with complexity O(N log N) and block error O(N^(−1/4)) under successive-cancellation decoding. The exact recursion on the erasure channel BEC(½) shows the polarization. With $N = 1024$ (10 levels of $Z \to 2Z - Z^2$, Z^2 from $Z = 0.5$), 33.59% of the channels have Z below 10^{-3} and 33.59% above $1 - 10^{-3}$, while the mean Z stays at 0.5. The codes rest on Shannon 1948, which posed the problem: they are the first explicit construction provably reaching capacity, according to Wikipedia and Arıkan's abstract [39, 40]. No source states a dependency on Reed–Muller 1954, so no arrow is drawn. Polar codes were adopted for the control channels of 5G NR, in November 2016 according to Wikipedia [41, 40].

In 2016 Kudekar, Kumar, Mondelli, Pfister, Şaşoğlu and Urbanke proved that doubly transitive codes, and hence Reed–Muller and extended BCH codes, achieve capacity

on the erasure channel under MAP decoding [27, 42, 43, 44]. The proof uses sharp thresholds and the EXIT area theorem. It came 62 years after Reed–Muller. It settles the capacity conjecture for Reed–Muller codes on this channel, a conjecture "suggested previously in the literature". The paper appeared in IEEE Trans. IT 63 (2017) 4298–4316 and won the 2021 IT Society Paper Award [43, 44].

In 2023 Abbe and Sandon proved that Reed–Muller codes achieve Shannon capacity on symmetric channels with vanishing block error [47, 48]. The proof boosts recursively from a non-trivial base success probability. It received the 2025 IEEE IT Society Paper Award (note 9). It proves the Reed–Muller capacity conjecture. The abstract says that it re-proves the bit-error result of Reeves and Pfister more briefly, rather than using it [47].

In 2024 Abbe, Sandon, Shashkov and Viazovska gave a polarization theory for Reed–Muller codes, with vanishing local error below capacity [42, 49]. The proof uses entropy extraction and the Polynomial Freiman–Ruzsa theorem. The paper appeared on arXiv on 20 November 2024 (v2 on 25 February 2026) and is a preprint as of 28 October 2026. It uses Arıkan's polarization, the PFR theorem and Abbe–Sandon 2023, which its first abstract cites as [AS23], calling the new work a second proof of vanishing local error (note 10).

Notes

1. Fano's independence is settled from Shannon's 1948 text ("found independently by R. M. Fano") and Wikipedia's Shannon–Fano article [5, 18].
2. Shannon's classified cryptography report is dated 1 September 1945 by historyofinformation.com, Wikipedia, Diffie (IACR) and the memorandum's own cover sheet (MM-45-110-92 in the scan; Wikipedia writes MM 45-110-02) [11, 12, 13, 14]. Footnote 1 of the York transcription of the 1949 paper gives 1 September 1946 [10]. The date is settled as 1945.
3. Only Wikipedia states Redheffer's role in Kraft's thesis [19].
4. Costello and Forney say only that Golay's work came "shortly after" Shannon's; no source consulted states that Golay drew on Shannon [17].
5. The Reed–Solomon paper cites Hamming as BSTJ vol. 26, but the archive.org record gives vol. 29, no. 2, April 1950, pp. 147–160, and the Utah BSTJ 1950 contents agree [53, 54]. This settles it.
6. That Reed–Solomon rests on Hamming 1950 and Reed 1954 is stated only in the paper's own reference list [26].
7. Tandon's dependencies, and his result itself, are stated only in his own preprint; the only other page found reproduces the arXiv abstract [52]. Every 2026 bound on $\Theta(C_7)$ is a preprint, and a newer bound than Tandon's may exist, though none was found on 28 October 2026.
8. The patent dates are taken from the Google Patents record [37]. Wikipedia's turbo-code article gives filing as 23 April 1991 (the French priority date) and expiry as 29 August 2013 [36]; for a US patent filed before June 1995 the term ends 17 years after grant, which is 29 August 2012. The patent naming Berrou alone, and the paper naming three authors, is not a factual conflict.
9. The venue of Abbe–Sandon 2023 is not confirmed; the arXiv record gives no journal reference [47].
10. The links from Abbe–Sandon–Shashkov–Viazovska to Arıkan, to the PFR theorem and to Abbe–Sandon 2023 are stated only in the paper itself; the NCCR SwissMAP

page repeats its summary [42, 49].

11. Not researched for this chapter: Singleton, Berlekamp–Massey, Elias, Forney, Richardson–Urbanke and Guruswami–Sudan. The MSC classes covered are 94A Communication and information, 94B Error-correcting and error-detecting codes, 94C Circuits and networks, and 94D, whose MSC2020 title is “Miscellaneous topics in information and communication theory” [55, 56]. IMPAN’s wording “fuzzy sets and logic” names the subclass 94D05, not the class.

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Mathematics education

This chain is not a chain of proofs. Its links are textbooks, reforms and comparative studies, and an arrow means that a reform builds on or reacts to an earlier one. It starts with Euclid's *Elements*, the teaching text for about 2000 years and first printed in 1482. It runs through Klein's Meran reform (1905) and the founding of ICMI (1908) to the New Math that followed Sputnik, against which Freudenthal (1971) and Kline (1973) reacted.

The chain ends in two large comparative studies, TIMSS (1995) and PISA (2000). In 2026 the PISA 2025 results have been published, and TIMSS 2027 is scheduled. No source documents an arrow from TIMSS to PISA: they are parallel programmes. In all, the chain has sixteen links, from the *Elements* of about 300 BCE to TIMSS 2027.

Euclid as the school text

About 300 BCE Euclid wrote the *Elements* in thirteen books, an axiomatic and deductive presentation of geometry and number. MacTutor calls it "the centre of mathematical teaching for 2000 years" [1], and Britannica "a standard for deductive reasoning and geometric instruction" [2]. This is the first link; it rests on nothing earlier in the chain.

The text reached print in 1482, when Erhard Ratdolt published Campanus's Latin translation in Venice [1, 3, 4]. More than a thousand editions have followed. From about 300 BCE to 1482 is 1781 years, counting with no year 0. This link rests on the *Elements* through the transmission of the text. Britannica's dates of 1505 and 1533 are later firsts: the first Latin translation from the Greek and the Greek first edition (note 1).

Klein and the first international reform

In 1872 Felix Klein set out the Erlangen Programme: geometry as the study of the invariants of a group of transformations [5, 6]. The ICMI history describes it as "generalising the study of geometrical transformations" [5]. His inaugural address at Erlangen also set out an educational programme, sometimes called a "second Erlangen program" [5]. MacTutor adds that the Programme was written for the occasion but was not the speech he actually gave [6].

Thirty-three years later, in 1905, Klein and a commission of the German Mathematical Society (DMV) drew up the Meran syllabus. Its principle was that "the function concept should pervade all parts of the mathematics curriculum", with the rudiments of calculus taught in secondary school [5, 6]. A Würzburg history of German mathematics teaching confirms that the Meran guidelines, on functional thinking, the genetic structure of teaching and the joining of pure and applied mathematics, were influenced by Klein [35]. That the Erlangen Programme itself shaped Meran is an influence, not a dependency, and only the ICMI history suggests it (note 2).

Three years after Meran, in 1908, the International Commission on Mathematical Instruction (IMUK, later ICMI) was founded at the fourth International Congress of Mathematicians in Rome, with Klein at its head [5, 6] (note 3). National reports on mathematics teaching followed. In 1914, in Paris, the commission examined “the position occupied by calculus in the secondary schools” [5, 7]. ICMI rests on Meran through its reformer, Klein himself. That the 1914 session followed from Meran is inferred: Howson, who reports the session, does not join the two [7].

The New Math and its critics

In 1958, one year after Sputnik, Edward Begle founded the School Mathematics Study Group (SMSG) after two conferences funded by the National Science Foundation [8, 10]. It was a national curriculum project and became the core of the American New Math. Its trigger was external, not mathematical: Sputnik and NSF grants [10, 9]. Wikipedia also names the National Defense Education Act of 1958 as a source of money [9]. A chronology by the mathematician Ralph Raimi links NDEA funds to NSF science-education grants and calls SMSG the best financed NSF project, but it does not say that NDEA money paid for SMSG [36] (note 4).

From about 1958 to 1973 the New Math spread internationally. It taught set theory, logic and abstract algebra, and treated geometry through linear algebra. In France it was inspired by Bourbaki, and Jean Dieudonné’s slogan “Down with Euclid!” is often quoted [9]. The New Math reacted against Euclid’s *Elements*. Furinghetti, Matos and Menghini confirm that at the Royaumont seminar of 1959 Dieudonné held that Euclidean geometry could be dealt with in three lessons [12] (note 5).

Alongside these reforms ran the first international measurement. In 1964 the IEA carried out FIMS, its first international mathematics study, in twelve school systems [13, 14]. It was not the IEA’s first study: it grew out of the Pilot Twelve-Country Study of 1960, which had already tested 13-year-olds in mathematics [13]. The count of twelve comes from the IEA history and the Gothenburg COMPEAT centre, which lists all twelve; Wikipedia gives eleven countries [15] (note 6).

In 1971 Hans Freudenthal began Realistic Mathematics Education. It treats mathematics as a human activity, with “mathematizing” among its central ideas [16, 19]. The institute he founded that year, IOWO, later became the Freudenthal Institute, and he had founded the journal *Educational Studies in Mathematics* (*ESM*) in 1968 [16, 20]. His work kept the Netherlands out of the New Math; the ICMI history calls him “the energetic opponent of the structuralist ‘New Math’ ” [16, 20, 11]. This link reacts against the New Math. It also reacts to FIMS: Freudenthal criticised it in “Pupils’ Achievements Internationally Compared – the IEA” (*ESM* 6(2), 1975), and the IEA replied through Peaker in *ESM* 7 (1976) [17, 18, 16].

In 1973, fifteen years after SMSG began, Morris Kline published *Why Johnny Can’t Add: The Failure of the New Math* (St. Martin’s Press) [21, 22]. The book is a critique of the New Math. Wikipedia quotes its central claim: “abstraction is not the first stage, but the last stage, in a mathematical development” [9] (note 7).

The comparative studies

In 1995 the IEA ran the first cycle of TIMSS, repeated every four years, and trend monitoring began that year [23]. The first cycle covered 46 participating countries

according to the IEA history [13], or 41 educational systems according to Wikipedia [15] (note 6). The four-year cycle gives 1995, 1999, 2003, 2007, 2011, 2015, 2019, 2023 and 2027. Wikipedia names FIMS of 1964 as the predecessor [15]. The IEA history places the 1960 pilot, FIMS, SIMS (1980–82) and TIMSS 1995 in one line of studies, but does not call FIMS the predecessor [13]; the dependency is inferred from that line.

In 2000, five years after TIMSS, the OECD ran the first cycle of PISA, repeated every three years [24, 26]. It tests whether 15-year-olds can use mathematics, reading and science in real life [25, 24]. PISA does not rest on TIMSS. It is a parallel programme run by a different organiser.

The PISA 2022 results were released on 5 December 2023. The round was “originally scheduled for 2021 and postponed” because of COVID-19 [27]. The OECD mean in mathematics fell by “almost 15 score points” from 2018 [27]; Euronews called it “a record 15 points” [28], and Al Jazeera also gives 15 [26]. Nearly 700,000 students in 81 systems took part, and Singapore came top in all domains [28, 24]. Wikipedia’s means, 489 in 2018 and 472 in 2022, give a fall of 17 points, not 15 (a quick check: $472 - 489 = -17$); the sources do not explain the gap (note 8). This link rests on the same PISA instrument, repeated.

The TIMSS 2023 results, the eighth cycle, were released on 4 December 2024 [30]. In grade 8 mathematics Singapore led with 605, followed by Chinese Taipei with 602 and Korea with 596 [29, 15]. At grade 4 Singapore scored 615 [29]. The link rests on the same TIMSS instrument, repeated.

The PISA 2025 results were published on 8 September 2026 [31, 34, 33]. They cover 91 systems and more than 760,000 students; these figures come from the Swedish OECD delegation and Teacher magazine, as the OECD publication page gives neither [32, 33]. Science was the major domain [34]. Beijing–Shanghai–Jiangsu–Zhejiang led in mathematics with 612, and Singapore came second with 563 [33, 24]. The link rests on PISA 2022 through the same instrument.

The last link lies ahead. The IEA states that “the next cycle of TIMSS will be administered in 2027” [23, 15], and it published the TIMSS 2027 assessment frameworks on 9 September 2025 [37]. The date fits the four-year cycle: $1995 + 8 \times 4 = 2027$. It rests on TIMSS 2023 through the same instrument.

Notes

1. The first printed *Elements* is Ratdolt’s Venice edition of 1482 [3, 4, 1]. Britannica’s 1505 and 1533 are later firsts: the first Latin version from the Greek, and the Greek text [2]. The MAA calls the 1482 text Campanus’s translation [3]; historyofinformation.com calls it Adelard’s translation edited by Campanus [4]. Both descriptions are kept.
2. Only the ICMI history suggests that the Erlangen Programme influenced the Meran syllabus [5]. No source states that the 1914 Paris session followed from Meran.
3. The ICMI history and Howson call Klein the commission’s “president” [5, 7]; MacTutor calls him its “chairman” [6]. Both are kept.
4. Only Wikipedia states that NDEA money funded SMSG [9]. Raimi’s chronology supports the link to NSF grants generally [36].

5. Wikipedia gives Dieudonné’s slogan as “Down with Euclid!” [9]; it is also quoted as “Euclid must go!”. No second source gives either wording, though Furinghetti, Matos and Menghini confirm the 1959 Royaumont lecture and its three-lesson Euclid [12].
6. Counts differ. TIMSS 1995 had 46 participating countries according to the IEA history [13] and 41 educational systems according to Wikipedia [15]; the sources do not explain the difference, so both are given. FIMS had twelve systems according to the IEA history and Gothenburg COMPEAT [13, 14], but eleven countries according to Wikipedia [15]; twelve is kept.
7. Only Wikipedia gives Kline’s quoted sentence [9]. The subtitle “The Failure of the New Math” follows the library catalogue record and MacTutor’s references [22, 21].
8. For PISA 2022 the OECD reports a fall of “almost 15 score points” [27], while Wikipedia’s means (489 to 472) give −17 [24]. The OECD figure is the headline.
9. Legendre’s *Éléments de géométrie* (1794), the British Association for the Improvement of Geometrical Teaching (1871), the Perry movement (1901) and the NCTM Standards (1989) are left out of the chain for lack of a source.

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General

This section has two parts, and a subsection on modelling. Philosophy of mathematics is a chain of positions and arguments, not of proofs. Plato, Aristotle and Kant lead to Frege's logicism, which Russell's paradox (1902) breaks. Brouwer's intuitionism provokes Hilbert's programme, which Gödel (1931) shows cannot succeed. Benacerraf (1965) motivates structuralism, and univalent foundations (2013) are the current frontier.

Recreational mathematics, by contrast, holds real theorems. Its chains run through the magic-square counts from Frénicle (1693) to order 6 (2024), Euler's Königsberg bridges (1735/1741) and Hierholzer's converse, and Conway's Game of Life from Gardner's column (1970) to omniperiodicity (a preprint, 2023). Mathematical modelling (00A71) has its own chain below. No source links the Chinese Lo Shu to Frénicle.

The class codes are those of the AMS MSC2020 list: 00-XX "General and overarching topics; collections", 00Axx "General and miscellaneous specific topics", 00A08 "Recreational mathematics", 00A30 "Philosophy of mathematics" and 00A71 "General theory of mathematical modeling" [1]. IMPAN's MSC list gives the same three codes, with 00A71 as "Theory of mathematical modeling" [2] (note 1). In all, the first two parts hold 29 links, from Plato to univalent foundations and from the Lo Shu to omniperiodic Life.

Philosophy of mathematics: from the forms to the paradox

Plato, teaching at the Academy from about 387 BC to his death in 347 BC, held that mathematical objects are perfect forms. Drawn diagrams are only "imperfect imitations": a line has length but no breadth, and no physical line does [3, 4]. This is the first link of the chain.

Aristotle, in the fourth century BC, rejected separate mathematical objects. For him they are perceptible magnitudes studied "qua" quantity, and the infinite exists only potentially: it is "always possible to take something outside" [4, 5]. His view is a critique of Plato; MacTutor says that "he did not agree with Plato", and that the infinite "only potentially exists" [5].

In the *Critique of Pure Reason* (1781; second edition 1787) Kant argued that mathematical judgments are synthetic a priori [6, 8, 9]. They come from construction in pure intuition: analysing the concepts 7 and 5 never yields 12, so counting is needed [6, 7]. No earlier link is named as its basis.

Frege's logicism, set out in the *Grundlagen* (1884) and the *Grundgesetze* (1893 and 1903), derives arithmetic from pure logic [10, 11]. Numbers are defined through extensions of concepts, using his Basic Law V. This reacts against Kant: for Frege arithmetic is analytic, for Kant synthetic [12], and Frege rejected Kant's grounding of arithmetic in "pure intuition" [13].

In a letter of 16 June 1902 Russell told Frege of his paradox [10]. The set of all sets that are not members of themselves shows that Basic Law V entails a contradiction

[11]. The paradox is a counterexample to Basic Law V. Frege's repair in the 1903 appendix was later shown inconsistent as well: by Leśniewski in 1938, according to Freund (1982, after Sobociński), and independently by Quine in 1955 ("On Frege's way out", *Mind* 64) [14, 10]. MacTutor says only that this came "after Frege's death" [10].

Philosophy of mathematics: intuitionism, formalism and incompleteness

In 1907–1908 Brouwer founded intuitionism. Mathematics is mental construction; there is no completed infinite; the law of the excluded middle and non-constructive existence proofs are rejected [11]. According to the SEP, Brouwer was inspired by Kant [6, 11]; MacTutor links him to Poincaré instead [18] (note 2).

Whitehead and Russell's *Principia Mathematica* (1910–1913) offered type theory as a repaired logicist foundation [11]. It repairs the gap that the paradox exposed: MacTutor writes that "Russell's own response to the paradox came with the introduction of his theory of types" [16, 15].

In the early 1920s Hilbert proposed his programme: formalise mathematics and prove its consistency by finitary means [17]. It answers the criticisms of Brouwer's intuitionism and of Weyl; Zach writes that Hilbert set out "to answer Brouwer and Weyl's criticisms: the finitary point of view" [17, 19, 18].

In 1931 Gödel proved his incompleteness theorems. The proof arithmetises syntax, coding formulas and proofs by numbers, and builds a self-referential, diagonal sentence [19]. As a consequence "the consistency of the axioms cannot be proved" within the system [15]. *Principia* was his target system, and the result refutes Hilbert's programme: "Gödel's work is generally taken to show that Hilbert's Program cannot be carried out" [17].

In 1936 Gentzen proved the consistency of Peano arithmetic by transfinite induction up to ε_0, "Cantor's first epsilon number" [20, 17]. This is a partial, relativised repair that goes beyond finitary methods. It rests on Gödel's theorem: as MacTutor notes, the proof had to use tools stronger than those of the system [20, 11]. The SEP's general entry cites "Gentzen 1938", which is his second proof (note 3).

Philosophy of mathematics: after the foundational crisis

In 1965 Benacerraf published "What Numbers Could Not Be" [21, 11]. Zermelo's and von Neumann's numerals both model arithmetic, yet they differ, so numbers cannot be either set; what matters is structure. A quick check: in Zermelo's numerals $3 = \{\{\{\varnothing\}\}\}$ has one element, in von Neumann's $3 = \{0, 1, 2\}$ has three; and $1 \in 3$ is false in the first, true in the second.

In "Mathematical Truth" (1973) Benacerraf posed a dilemma. The natural semantics of mathematics needs abstract objects, but a causal theory of knowledge cannot reach them [11, 21]. Neither of his two papers rests on an earlier link of the chain.

The indispensability argument says that our best science quantifies over mathematical entities, so we should accept them. Quine alludes to it from "On What There Is" (1948) and "Two Dogmas of Empiricism" (1951) onwards. Putnam's *Philosophy of Logic* (1971) is its classic statement [23, 22, 11].

In *Science without Numbers* (1980) Hartry Field gave a nominalist reconstruction of Newtonian mechanics, which undercuts indispensability [11, 22, 23]. It is a counter to the indispensability argument. According to the SEP it is also an answer to Benacerraf's 1973 dilemma [11].

Neo-logicism returns to Frege. Wright (1983) derived the Peano axioms from Hume's Principle, which replaces Basic Law V, and Boolos (1987) showed the principle consistent; second-order Peano arithmetic follows, a result known as Frege's theorem [12, 11, 25]. The repair answers Russell's paradox. Hume's Principle is "one of the consistent principles that Frege discussed in 1884" [12], and Frege's theorem was rediscovered by Wright in the early 1980s [24]. The SEP records that "George Boolos and others observed that Hume's Principle is indeed consistent", and Hale and Wright cite Boolos's 1987 paper "The Consistency of Frege's Foundations of Arithmetic" [11, 25].

Structuralism was developed by Shapiro (1997) and by Resnik [26, 27, 11]. In the SEP's words: "The number three, for instance, will on this view not be an object but a place in the structure of the natural numbers" [11]. It rests on Benacerraf's 1965 paper, whose identification problem is "the main motivation for arriving at a structuralist understanding" [11, 21]. According to the SEP it also draws on Benacerraf's 1973 dilemma [11].

The current frontier is univalent foundations. In 2013 Voevodsky and the Univalent Foundations Program published the HoTT book [29, 28]. It adds to intensional Martin-Löf type theory the univalence axiom, "equivalent types are identical" (formulated in 2006–09), and higher inductive types, and proposes the result as an alternative to set theory. It rests on Martin-Löf type theory [29, 28, 30]. Its tie to structuralism is one of influence: Awodey (2013) notes that univalence lets one "identify isomorphic objects" and is "significant for … structuralism" [31, 28].

Recreational mathematics

The Lo Shu, the 3×3 magic square, is recorded in China in about 190 BCE, with legends from about 650 BCE [33, 32] (note 4). It is unique up to symmetry. A quick check: an exhaustive search finds exactly 8 magic squares of order 3, the 8 symmetries of one, such as the rows (2, 7, 6), (9, 5, 1), (4, 3, 8).

In 1514 Dürer put a 4×4 magic square into his engraving *Melencolia I*, with the date in the bottom row [32, 33]. Every row, column and diagonal sums to 34, and the middle cells of the bottom row read 15, 14 (checked).

In 1693, in a posthumous publication, Frénicle de Bessy showed by exhaustive classification that there are exactly 880 normal magic squares of order 4 [32, 33]. A full computer enumeration confirms it: there are 7040 such squares, and 7040 / 8 = 880 up to rotation and reflection. Frénicle's work does not rest on the Lo Shu; no source links the two, and they belong to parallel traditions.

Graph theory began with a puzzle. Euler's paper on the bridges of Königsberg (E53) was presented in 1735 and printed in 1741 [34, 35] (note 5). Every vertex of a walk other than its two ends needs even degree. Königsberg has four vertices of odd degree, but a walk has at most two ends, so no walk crosses each bridge once. A quick check: the degrees are 5, 3, 3, 3, and a brute-force search finds no such walk.

In 1873 Hierholzer proved the converse: a connected graph with 0 or 2 vertices of

odd degree has such a walk [35]. His paper appeared in *Mathematische Annalen* 6 (1873), 30–32, “on the possibility of traversing a line-system without repetition or interruption” [36]. Euler stated the converse, and Hierholzer proved it [35, 36].

The Game of Life, with rule B3/S23, was devised by John Conway and published in Martin Gardner’s “Mathematical Games” column in *Scientific American* in October 1970 [37, 38, 39] (note 6). Conway offered $50 for an answer to whether a pattern can grow without limit [38, 39]. A quick check by simulation: the glider {(1,0), (2,1), (0,2), (1,2), (2,2)} after 4 generations equals itself shifted by (1, 1).

In November 1970 an MIT team led by Bill Gosper found the glider gun, which emits a glider every 30 generations, so unbounded growth exists [38, 39]. This solves Conway’s question, and the team won the $50 prize [39].

In 1973 Richard Schroeppel counted 275,305,224 normal magic squares of order 5 by computer enumeration [32, 40, 41]. This count was not recomputed here. It continues the same counting problem at the next order.

In *Winning Ways* (1982) Berlekamp, Conway and Guy showed that Life contains a universal computer and a universal constructor, built from logic gates made of glider streams [42, 43, 38]. LifeWiki dates the proof to 1982, and Wikipedia cites the second edition of 2001–04 [42, 38]. No earlier link is named as its basis.

In 1996 David Buckingham showed, in “My Experience with B-heptominos in Oscillators”, that Life has oscillators of every period from some threshold on [44, 45]. His method used Herschel-conduit constructions. Quanta gives the threshold as 61, LifeWiki as 58 for oscillators and 62 for guns (note 7).

For order 6, Pinn and Wieczerkowski (1998) estimated the number of magic squares by Monte Carlo as $0.17745(16) \times 10^{20}$ [46]. In May 2024 Hidetoshi Mino presented the exact count, 17,753,889,197,660,635,632 [40, 32]. The two agree: exact / estimate = 1.0005, within 0.56 of the estimate’s standard error (recomputed). This continues the same counting problem (note 8).

In 2023 Brown, Cheng, Jacobi, Karpovich, Merzenich, Raucci and Riley showed that Life is omniperiodic: it has oscillators of every period [47, 44, 45]. The last two missing periods were 19 and 41, found by computer search plus constructions. The result builds on Buckingham-type constructions and on the Snark loop (2013) for periods from 43 on, and Conway’s rules posed the question. As of 29 October 2026 it is a preprint (arXiv 2312.02799, version 1 of 5 December 2023, with no journal reference) [47].

Mathematical modelling (00A71): dimensional analysis

This chain covers the method of modelling, not any one model, and runs in three strands. Dimensional analysis leads from Galileo (1623), Newton’s similarity proposition (1687) and Fourier’s homogeneity (1822) to the π theorem of Vaschy (1892), Federman and Riabouchinsky (1911) and Buckingham (1914). The philosophy of fit runs from Wigner (1960) to Box (1976/1979). Model selection and verification and validation run from Akaike (1973) and Schwarz (1978) to the SCS terms (1979), Oreskes and colleagues (1994) and the AIAA guide G-077 (1998).

Most arrows in this chain are influence, rivalry or shared definitions, not proof dependencies. The proof dependencies are rank–nullity → π theorem, Kullback–

Leibler → AIC, Laplace asymptotics → BIC, and Buckingham → Hardtke (2019) and IT-π (2025). In 2026 the new work is all in dimensional analysis: IT-π (*Nature Communications*, 2025) and two 2026 preprints. No 2025–2026 result was found in the Wigner–Box or V&V strands.

Several links are independent of their neighbours. Newton does not rest on Galileo, nor Fourier on Newton; Vaschy, Federman, Riabouchinsky and Buckingham found the π theorem independently; and BIC is a rival to AIC, not a step in its proof. The chain has 23 links, from Galileo to Valorani.

Galileo's contribution is a programme, not a theorem. In *Il Saggiatore* (1623) he wrote that the book of nature "is written in the language of mathematics" [48, 49]. He stated laws as proportions and tested them on inclined planes, and in 1638 gave the law that the distance of fall grows as t^2. From 1623 to the *Principia* (1687) is 64 years.

In 1687 Newton proved, in Proposition 32 of Book II of the *Principia*, that similar systems of particles, started similarly, move similarly in proportional times [50, 51]. Scaling lengths, times and densities keeps the motion similar. This is the first scaling law. No source states that it rests on Galileo.

In 1822 Fourier gave every quantity an exponent of dimension in length, time and temperature, found by varying the units [52, 53, 54, 55]. Every term of a law must carry the same exponents: the principle of homogeneity. He also gave the first table of dimensions. A quick check: K/(CD), conductivity over specific heat times density, has dimension L^2T^{-1}, and both $\partial v/\partial t$ and $(K/CD)\partial^2 v/\partial x^2$ have exponents [0, 0, −1, 1] in (M, L, T, Θ), so the heat equation is homogeneous. No source says Fourier used Newton's Proposition 32; the two are independent.

By 1871 Maxwell, with Fleeming Jenkin, had made mass, length and time the fundamental units and written derived units as products of powers of M, L and T, the modern notation [55, 54]. Rivadulla quotes Maxwell in 1871: M, L and T "are the units of mass, length and time" [54]. He also dates Maxwell's first use of dimensional notation to the 1863 paper with Jenkin, "On the elementary relations of electrical quantities" [54]. Mitchell (2017) credits Maxwell, Thomson and Jenkin with "the invention of the dimensional formula" [56]. Maxwell built on Fourier: in 1877 he wrote that the theory of dimensions was "first formulated by Fourier", and in an 1870 letter that the first table of dimensions he knew was Fourier's [54, 55].

In *The Theory of Sound* (1877) Rayleigh used the method of dimensions [57, 55]. One writes the unknown as a product of powers of the governing quantities and solves the linear equations in the exponents (note 9).

According to Wikipedia, Joseph Bertrand proved the π theorem for special cases in 1878 (*Comptes rendus* 86), in electrodynamics and heat conduction, by applying homogeneity to particular formulas [58]. Sterrett confirms only that Bertrand's related work predates 1914, and Görtler does not name him [51, 61] (note 10). From 1878 to Buckingham is 36 years.

In 1892 Aimé Vaschy gave the first general π theorem, for any number of quantities, in the *Annales télégraphiques* 19 [59, 61, 57]. Hardtke calls it "essentially the same result" as Buckingham's [59]. Görtler dates Vaschy's first formulation to 1890, in his *Traité d'électricité et de magnétisme* [61] (note 10). This is 22 years before Buckingham, or 24 counted from 1890. No source says whether Vaschy knew Bertrand's cases.

In 1911 the general π theorem was found twice more, independently: by A. Federman in the *Izvestiya* of the St Petersburg Polytechnic Institute (16), and by Dimitri Riabouchinsky in his “variables de dimension zéro” in *L’Aérophile* [59, 58, 57]. Neither is a source for Buckingham; they were published independently.

In 1914 Edgar Buckingham published the π theorem in the *Physical Review* 4, 345–376, and named it with the symbol π [58, 59, 57]. A law among n quantities that does not depend on the choice of units, whose dimension matrix A has rank r, is a law among n − r dimensionless products. The proof uses the fact that unit changes act linearly in logarithmic coordinates. The invariants are the products q^x with Ax = 0, and by rank–nullity ker A has dimension n − r.

Two quick checks. For the pendulum (period T, mass m, length l, gravity g) the rank is 3, so there is one group, π = gT²/l. With T = 2π√(l/g) it equals 39.478418 (that is, 4π²) at (l, g) = (1.0 m, 9.81 m/s²), (2.5 m, 9.81 m/s²) and (0.7 m, 3.71 m/s²). For the drag on a sphere (F, ρ, μ, D, V in M, L, T) the rank is 3, so there are 5 − 3 = two groups.

Buckingham’s theorem uses rank–nullity in its proof. Fourier’s homogeneity is its hypothesis; a recent paper on augmented dimensional analysis counts Fourier among the contributors to the theorem’s history, with Vaschy and Federman [86]. Rayleigh’s method was an influence that Buckingham acknowledged.

In 1915 Rayleigh published “The Principle of Similitude” in *Nature* 95, 66–68 [62]. It is a plea for the method of dimensions, with a heat-transfer example worked by matching exponents. It rests on his method of 1877. Riabouchinsky objected that Rayleigh had treated heat, temperature, length and time as four independent units; with only three, the result is less definite [54].

In 1922 Percy Bridgman published *Dimensional Analysis*, usually called the first complete treatise and a general theory [61, 54, 55]. It systematised the π theorem. No source states an arrow to it (note 11). From 1890, the date of Muñoz del Castillo’s rival claim, to 1922 is 32 years, as Rivadulla says.

Mathematical modelling: fit, selection and validation

In February 1960 Eugene Wigner published “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” in *Communications on Pure and Applied Mathematics* 13, No. 1 [63, 64]. It is not a theorem but an argument from examples: gravitation, helium, the Lamb shift. Wigner notes that Newton’s law, first checked to “about 4%”, now holds to “less than a ten thousandth of a per cent” (10^{-4} %), a factor of at least 4 / 10^{-4} = 40,000. The lecture was given on 11 May 1959 [64]; to the February 1960 issue, whose day is not given, is 266 days to 1 February and 294 days to 29 February.

Wigner uses Newton’s gravitation as evidence, an influence only and not Proposition 32. Richard Hamming replied in 1980 in the *American Mathematical Monthly* 87(2): “The inspiration for this article came from the similarly entitled article” by Wigner [65, 64].

In 1973 Hirotugu Akaike published his information criterion, AIC = −2 ln L + 2k [68, 66, 67]. It estimates the relative Kullback–Leibler information lost by a model [67]. It was announced in 1971, printed in 1973 and formalised in 1974 in the *IEEE Transactions on Automatic Control* 19 [67, 68]. The proof uses the Kullback–Leibler divergence of 1951.

A quick simulation shows how AIC behaves. Take numpy's default generator with seed 0, 60 equally spaced points on $[-1, 1]$, and $y = 1 + 2x - 3x^2$ plus normal noise with standard deviation 0.5. Fit least-squares polynomials of degree 0 to 6, with k = degree + 2 (coefficients and variance). AIC is lowest at degree 3 (73.79, against 80.07 at the true degree 2). Over seeds 0–999 AIC picks a degree above 2 in 27.6% of runs.

In 1976 George Box wrote, in "Science and Statistics" (*JASA* 71): "Since all models are wrong the scientist cannot obtain a 'correct' one by excessive elaboration" [69, 70]. It is a methodological thesis on parsimony and on iterating between theory and data. It shares the principle of parsimony with AIC and BIC, but no source shows a citation link.

Three years later, in 1979, Box gave the full aphorism "All models are wrong, but some are useful" [71, 70, 72]. It appears as a section heading, "ALL MODELS ARE WRONG BUT SOME ARE USEFUL", on page 202 of his 1979 chapter (Academic Press) [72]. The 1987 wording by Box and Draper (p. 424) begins "Essentially" [71]. This link rests on Box's own 1976 paper.

In 1978 Gideon Schwarz introduced the Bayesian information criterion in the *Annals of Statistics* 6 [73, 74]. One chooses the model that maximises $\log M_j - \tfrac{1}{2} k_j \log n$, a rule independent of the prior. The proof expands the posterior integral asymptotically, a Laplace-type expansion, which Schwarz proves with three lemmas of his own [73]. AIC is a rival criterion, not a step in the proof.

The BIC penalty $k \ln n$ exceeds AIC's $2k$ when n is above $e^2 = 7.389$, so from $n = 8$ on; Schwarz says "when there are 8 or more observations" [73]. With the same grid and parameters as above, seed 0 gives BIC lowest at degree 3 (84.26, against 88.45 at degree 2). Over seeds 0–999 BIC picks the true degree 2 in 92.9% of runs, against 72.4% for AIC, and a higher degree in 7.1%, against 27.6% (all recomputed).

In 1979 the Technical Committee on Model Credibility of the Society for Computer Simulation, chaired by Schlesinger, published "Terminology for model credibility" [75]. It distinguishes reality, the conceptual model and the computerised model, and defines verification and validation. It is definitional. Sargent (2005) adopts its definition of validation, "the definition used here", though not its definition of verification, and does not call it a standard [76]. From 1979 to the AIAA guide is 19 years.

In 1994 Naomi Oreskes, Kristin Shrader-Frechette and Kenneth Belitz argued in *Science* 263 that verification and validation of models of natural systems is impossible [77]. Such systems are never closed and model results are not unique, so confirmation is only partial. The argument is logical. It was published 32.73 years before 28 October 2026. Sornette and colleagues (2007) argue against it [78].

In 1998 the AIAA published its guide G-077-1998 [79, 80]. Verification asks whether a simulation represents the conceptual model; validation asks whether it represents the real world [79]. NASA's NPARC alliance follows these guidelines [80]. The guide states that it "builds primarily on terminology established by the Society for Computer Simulation and the Defense Modeling and Simulation Office" [79].

In 2019 Hardtke gave a quantitative, unit-free version of Buckingham's theorem [60, 59]. A function F in a scaling class $S(A, \beta)$ is $C \cdot v^{y}$ when A has full rank and $G(\pi) \cdot v^{y}$ otherwise, with π taken from a basis of the kernel of A. The proof uses the Buckingham π theorem, as the paper itself states, and it also draws on Curtis, Logan and Parker (1982) [59]. It is a preprint (checked 28 October 2026).

In 2025 Yuan and Lozano-Durán published IT-π in *Nature Communications* 16, 9171 [81, 82, 83]. Among the dimensionless groups the π theorem allows, the method chooses those that carry most information about the output. An irreducible-error theorem bounds what any model can achieve. It uses the π theorem: MIT's abstract speaks of dimensionless variables "as guided by the Buckingham-π theorem" [83]. It came 111 years after Buckingham (note 12).

In 2026 Habera and Zilian automated dimensional analysis and nondimensionalisation in the finite-element language UFL [84]. Dimensions are treated as vectors in $\mathbb{Q}^n$, and nondimensionalisation as operator preconditioning. The method uses Maxwell-style exponent vectors. It is a preprint (arXiv version 2 of 15 June 2026, with no journal reference; checked 29 October 2026).

Also in 2026, Valorani found π groups from data alone [85]. The method takes logarithms, uses a singular value decomposition to find the manifold, and searches for integer exponents; it reproduces a compressor map with under 0.01% error. There is no new proof. It builds on the Buckingham π theorem, according to the paper itself. It is a preprint (arXiv version 1 of 15 June 2026, with no journal reference; checked 29 October 2026).

Notes

1. The AMS MSC2020 list names the class "General and overarching topics; collections" [1]; Wikipedia uses the MSC2010 name "General". The AMS is followed.
2. Only the SEP links Brouwer to Kant [6, 11]; MacTutor names Poincaré [18]. Only the SEP links Field to Benacerraf's 1973 paper, and that paper to structuralism [11].
3. Gentzen's first consistency proof is of 1936 [20, 17]; the 1938 date in the SEP's general entry [11] is taken to be his second proof. Both are reported.
4. The Lo Shu is "said to be written in 190 BCE", with legends from about 650 BCE, and its link to the Luoshu chart appears only in the twelfth century [33]. MathWorld gives no date [32].
5. Euler's Königsberg paper was presented on 26 August 1735, is often cited as 1736, and was printed in 1741 [34, 35].
6. Quanta says the rules of Life were devised in 1969 [44], Wikipedia 1970 with experiments from 1968 [38], and MacTutor "shortly after" 1968 [37]. The October 1970 publication has two sources [38, 39].
7. Buckingham's 1996 threshold is 61 in Quanta [44] and 58 for oscillators (62 for guns) in LifeWiki [45]. The difference is unsettled.
8. MathWorld and trump.de give the exact order-6 count [32, 40], which agrees with the 1998 estimate. Wikipedia still says the number is "not yet known for orders beyond 5 and 6" [33]; its wording is the odd one out.
9. Rayleigh's method is dated 1872 (first use, Wikipedia and Pesic), 1877 (publication, all sources) and 1899 (Taylor and colleagues).
10. Wikipedia says Bertrand (1878) first proved the π theorem [58]; Hardtke and MIT's course notes give Vaschy (1892) as the first general version [59, 57]; Taylor and colleagues cite Morozov (1908). The general statement predates Buckingham by 22 years; the Bertrand and Morozov claims have one source each. Görtler dates Vaschy's formulation to 1890, with the 1892 paper as a further reference [61]; this is unsettled. Görtler also says that Buckingham's

count p = n − m is wrong in general and that Bridgman's book stressed this, which fits the rank-based form n − r used here [61].

11. Bridgman (1922) is usually called the first treatise, but Rivadulla argues for José Muñoz del Castillo's *Unidades físicas* (1890), 32 years earlier [54]. Wikipedia dates dimensional analysis to Fourier (1822) [55], Sterrett traces physically similar systems to Newton [51], and Rivadulla the word "dimension" to Foncenex (1761) [54]; these are partly different claims.
12. The *Nature Communications* publication of IT-π is reported by the authors' institutions [82, 83]; the arXiv record (version 3 of 29 September 2025) lists no journal reference [81]. Taylor and colleagues place Buckingham's theorem in *Nature* in their text but cite the *Physical Review* (1914) in their references, as all other sources do. Some sources spell the 2019 author "Hartke"; arXiv's "Hardtke" is followed. Roache, the ASME V&V 10 and 20 standards, Oberkampf and Trucano (2002), AICc and TIC are not covered.

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History of mathematics

This chain is not a chain of proofs. Its links are works of historiography, and an arrow means that later work uses, transmits or corrects earlier work. It starts with Eudemus' lost *History of Geometry*, known through Proclus. It runs from Montucla (1758) and Moritz Cantor to Heath, who worked from Heiberg's texts.

Neugebauer opened up Babylonian mathematics. His reading of the tablet Plimpton 322 was later criticised by Robson (2001–02) and reread by Høyrup (2002). In 2026 the field rests on digital infrastructure: zbMATH Open (2021) and the MacTutor archive (1994). No source documents an arrow from Montucla to Cantor, and no milestone of 2025–2026 was found. The chain has seventeen links, from Plimpton 322 (about 1800 BC) to zbMATH Open (note 1).

The object and the ancient historians

The first link is not a history but the object that historians would later read. About 1800 BC an Old Babylonian scribe wrote the tablet now called Plimpton 322 [1, 2] (note 2). It holds fifteen rows of data on right triangles, the short side and the diagonal, in base 60. A quick check: with the scribal errors corrected, every row gives a whole-number long side that is a regular (5-smooth) number. Row 1 has short side 119 and diagonal 169, so the long side is 120, and $(d/l)^2 = 1;59,0,15$ in base 60, as Wikipedia gives. The first column decreases strictly down the tablet.

Before about 300 BC Eudemus of Rhodes wrote a *History of Geometry*, now lost [3, 4]. It is the main source for geometry before Euclid: for Thales, the Pythagoreans, Oenopides and Hippocrates of Chios [4]. Hippocrates' work on lunes is known only through it [3]. MacTutor gives Eudemus' life as about 350 to about 290 BC, and Wikipedia says he flourished before 300 BC [3, 4] (note 3). No earlier link is named as its basis.

In the fifth century AD Proclus (411–485) wrote his *Commentary on the First Book of Euclid's Elements*. MacTutor calls it "our principal source about the early history of Greek geometry" [5]. It rests on Eudemus, whose *History* Proclus "certainly used" [5, 4]. From about 300 BCE to about 450 CE, with no year 0, is $300 + 450 - 1 = 749$ years; both endpoints are approximate.

The great modern histories

In 1758 Jean-Étienne Montucla published his *Histoire des mathématiques*, "the first classical history of mathematics" [6, 7]. It was a history of ideas and problems, not lists of names. The second edition appeared as volumes 1–2 in August 1799 and volumes 3–4, completed by Lalande and others, in 1802 [7, 6]. From 1758 to Cantor's first volume is 122 years. No earlier link is named as its basis.

From 1880 to 1908 Moritz Cantor published his *Vorlesungen über Geschichte der Mathematik*, four volumes reaching the year 1799 [8]. The fourth volume (1908) had nine collaborators besides Cantor, who wrote one chapter [10, 8]. From volume 1

to volume 4 is 28 years. Montucla served as a model, not as a dependency. In his lecture at the 1900 International Congress, Cantor called him “a model that every historiographer of sciences must follow” [9], and the DSB says that “Montucla had no successor until Moritz Cantor” [7].

From 1884 to 1914 Gustaf Eneström edited the journal *Bibliotheca Mathematica*, which printed corrections to the standard history; he also compiled the Eneström index of Euler’s works [11]. According to Wikipedia he was a critic of Cantor’s work [11]. Smith (1909) confirms that *Bibliotheca Mathematica* printed “every month a list of corrections”, but he does not name Eneström [10] (note 4).

In 1906 Johan Ludvig Heiberg found the Archimedes palimpsest in Constantinople, with the text of *The Method* [13]. The title of Heath’s 1912 edition speaks of the work as “recently discovered by Heiberg” [14]. Heiberg published his transcriptions in 1910–15, in a complete edition of Archimedes [13]. No earlier link is named as its basis.

From 1908 to 1921 Thomas Heath published his English *Euclid* (1908, second edition 1926), *The Method* as a supplement (1912) and *A History of Greek Mathematics* (1921), all from the Greek editions [12, 13]. The 1912 supplement rests on Heiberg’s text of *The Method* [12, 14].

Babylonian mathematics and its readers

In 1931 Otto Neugebauer founded *Zentralblatt für Mathematik* and edited its first issue; in 1940 he also founded *Mathematical Reviews* [15, 16]. Zentralblatt had “essentially the same agenda” as the *Jahrbuch über die Fortschritte der Mathematik* (founded 1868) but was faster [17]. From the Jahrbuch to Zentralblatt is 63 years. Zentralblatt continues the Jahrbuch as an institution. It is the work of the same man as the cuneiform studies below, but it does not depend on them.

Neugebauer’s *Mathematische Keilschrift-Texte* (three volumes, 1935–37) was his own work [18, 15]. With Abraham Sachs he then published *Mathematical Cuneiform Texts* (1945) [18, 1]. They read Plimpton 322 as a table of Pythagorean triples generated from regular numbers p and q [2]. The square of column 3 minus the square of column 2 is a perfect square in every row, once four scribal errors are corrected [1] (note 5). This came 14 years after Heath’s *History* for the first work and 24 years for the second. A quick check confirms that $d^2 - b^2$ is a perfect square in all fifteen corrected rows. The link rests on Plimpton 322, now deciphered; it had earlier been read as a commercial record.

From 1949 to 1957 Evert Bruins proposed a reciprocal-pair reading of Plimpton 322: each triple is generated from a number x and its reciprocal 1/x [2, 19]. It is an alternative to the reading of Neugebauer and Sachs [2]. Robson’s abstract sets the reciprocal-pair reading, “first proposed about fifty years earlier”, against the generating-function theory [20]. Phillips writes that Robson’s reading “goes back to work of Bruins in 1949–1955” [19] (note 6).

In 1975 Sabetai Unguru published “On the need to rewrite the history of Greek mathematics” in the *Archive for History of Exact Sciences* 15(1), March 1975, pages 67–114 [33, 21, 22] (note 7). He rejected reading Greek geometry as “geometrical algebra”, the approach of Tannery, Zeuthen and van der Waerden [23, 24]. As quoted by Blåsjö, he held that “there has never been an algebra in the pre-Christian era”

[24]. His paper is a critique of the geometric-algebra readings, and Heath's essays stand for the algebraic approach: Acerbi treats them as representative, and Blåsjö cites Heath as the accessible statement of the traditional view [23, 24].

In 1994 John O'Connor and Edmund Robertson put the MacTutor History of Mathematics Archive on the web [25, 26]. It grew out of their HyperCard program "Mathematical MacTutor"; the site "went live in 1994" [26]. It now holds more than 3,200 biographies [27], and its creators received the Hirst Prize in 2015 [26]. No earlier link is named as its basis.

In 2001 and 2002 Eleanor Robson read Plimpton 322 as a teacher's list of regular reciprocal pairs [20, 2, 19]. Generating pairs, she argued, fail to explain the first column, the order of the rows and the errors, and the trigonometric reading is "conceptually anachronistic" [20]. This came 56 years after Neugebauer and Sachs. It is a critique of their generating functions (*Historia Mathematica* 28, 2001), and it takes up Bruins's reciprocal pairs [20, 19, 2].

In 2002 Jens Høyrup published *Lengths, Widths, Surfaces* [28, 29, 30]. By "conformal translation" he read Old Babylonian "algebra" as concrete cut-and-paste geometry [28, 30]. Melville reports that Høyrup distinguishes four groups of terms for what we call multiplication [34]; Wikipedia speaks of "two different additions and at least four different multiplications" [28]. Høyrup received the Kenneth O. May Prize in 2013 [28]. This came 57 years after *Mathematical Cuneiform Texts*. It rereads Neugebauer's material: Neugebauer saw a numerical algebra, Høyrup geometrical techniques [30], and Springer's description says that "the traditional interpretation must be rejected" [29].

The digital era

In 2011 Reviel Netz, Nigel Wilson and colleagues published *The Archimedes Palimpsest*, a diplomatic transcription made after multispectral imaging of the manuscript from 1999 to 2008 [13, 31]. It read the *Stomachion* as a work of combinatorics. It fills gaps in Heiberg's account: the project asked whether the manuscript held more "than had already been discovered by … Heiberg, 100 years earlier", and two Cambridge volumes had appeared by 2011 [31, 13].

In January 2021 FIZ Karlsruhe, the European Mathematical Society and the Heidelberg Academy made the full zbMATH database free as zbMATH Open [16, 17]. It includes the Jahrbuch records for 1868–1942, 223,276 JFM documents according to Teschke (Wikipedia rounds this to 200,000) [32, 17]. These records were digitised and incorporated in 2004 according to zbMATH, in 2003 according to Wikipedia [16, 17] (note 8). The database holds 4,866,444 documents [16]. This came 90 years after Zentralblatt, which it continues and integrates.

Notes

1. The AMS names the class "History and biography"; the msc2020.org front page gives "History of mathematics and mathematicians". The AMS wording is the class title.
2. Wikipedia dates Plimpton 322 to about 1800 BC (1822–1784 BC by its handwriting) [2]; MacTutor dates it between 1800 and 1650 BC [1]. Both are kept.

3. MacTutor gives Eudemus' dates as about 350 to about 290 BC [3]; Wikipedia puts his death around 300 BC [4]. Tannery held that Eudemus' *History* was lost before Pappus, Heiberg that Pappus and Eutocius still had it; only MacTutor reports this, and it is unsettled [3].
4. Only Wikipedia states Eneström's role as a critic of Cantor [11]; his specific corrections are not confirmed by a second source.
5. MacTutor reports four scribal errors in Plimpton 322, following Neugebauer and Sachs [1]; Wikipedia lists errors in rows 2, 8, 9, 13 and 15 [2]. The difference lies in counting and interpretation.
6. Phillips dates Bruins's work 1949–1955 [19]; the dates 1949–1957 are Wikipedia's [2].
7. Wikipedia gives the pages as 67–114 [21], as does Springer [33]; Acerbi gives 67–113 [23]. The replies to Unguru by van der Waerden, Freudenthal and Weil are known here only by title. Whether Heath's *History* (1921) used Proclus and Eudemus is not stated by any source used here.
8. Wikipedia gives zbMATH "more than 4 million records" [17], zbMATH itself 4,866,444 documents [16]; these are snapshots, and the zbMATH figure is used. zbMATH's own "some 200,000" counts books, not Jahrbuch entries [16]. The year in which the Jahrbuch was incorporated (2004 per zbMATH, 2003 per Wikipedia) is unsettled. Mansfield and Wildberger (2017) are left out of the chain.

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Where the chains meet

The chapters so far have followed one field at a time. This chapter collects the places where two chains meet: a crossing is a link from one field's chain whose proof, or whose stated basis, is a result from another field. Every crossing below already stands in one of the field chapters, with its sources there; the sources cited here are those of the link that states the crossing. A link whose dependency is "no arrow" is not a crossing.

In all there are 69 crossings, in 63 ordered pairs of fields, among 42 fields (a recount of the list confirms all three numbers). Geometry crosses with seven other fields, more than any other; number theory, algebra and analysis cross with six each. Unless a crossing is described otherwise, the later result uses the earlier one in its proof. The other kinds are named where they occur: a problem posed, a conjecture proved, a new proof, a codification, a generalisation, influence only, or background.

Numbers, foundations and algebra

From numbers and foundations to analysis: Cantor's theory of countable sets (1874) entered Lebesgue's integral (1902) through Borel's measure, whose countable additivity rests on it [1, 2, 3, 4]. From the same field to logic and computation: Cantor's diagonal argument (1891/92) is used by Russell's paradox (1902), by Gödel's incompleteness proof (1931), and by Church and Turing (1936) in their proofs that the decision problem is unsolvable [5, 6, 7, 8, 9, 10, 11, 12]. And to difference and functional equations: Hamel's counterexample of 1905, a non-linear solution of Cauchy's equation f(x + y) = f(x) + f(y), uses the axiom of choice (MSC 03E25) to build a basis of the reals over the rationals [13, 14].

From number theory to algebra the crossing is a problem posed. The failure of unique factorisation, and Kummer's ideal numbers that restored it, led Dedekind to his ideals of 1871 [15, 16]. In the other direction, from algebra to numbers and foundations, Bombelli's work on complex numbers (1572) uses Cardano's cubic formula of 1545: the formula produces square roots of negatives even when all three roots are real [45, 46, 47].

From algebra to unifying structures the crossing is a codification. Emmy Noether's ideal theory (1921) was codified in van der Waerden's *Moderne Algebra* (1930–31), and from there in Bourbaki's structures [48, 49, 50, 51]. From algebra to algebraic geometry, the rebuilding of the subject by Zariski (from 1935 into the 1940s) and Weil (1946) builds on the methods of *Moderne Algebra*; the sources describe this as a dependency of method rather than a single cited result (note 1) [52, 53, 54, 55, 56].

From analysis to numbers and foundations: Cantor's point sets of 1870–72, and set theory after them, grew out of Riemann's paper on trigonometric series and Weierstrass's limit points [60, 61].

Two crossings run from unifying structures to number theory. Deligne's proof of the Weil conjectures (1974) uses Grothendieck's étale cohomology and trace formula (1960–65), and Zhang's bounded gaps between primes (2013) uses bounds

of Deligne's type [101, 102, 103, 104, 105, 106, 107, 108, 109, 110]. Wiles's proof of Fermat's Last Theorem, with Taylor and Wiles (1995), starts from the Langlands–Tunnell theorem of the Langlands programme (begun 1967), which gives modularity mod 3 [111, 112].

From order and lattices to universal algebra: Birkhoff's general definition of an algebra (1933) builds on complete lattices, through the lattice of subalgebras (note 1) [113, 114, 115, 116].

Three crossings leave algebraic geometry. In 1983 Faltings proved Mordell's conjecture, a statement of number theory, with the methods of arithmetic algebraic geometry; this is a conjecture proved [117, 54, 118, 119]. Grothendieck's K_0 (1957) is a generalisation made in proving his version of Hirzebruch–Riemann–Roch (1954), and it began K-theory [120, 121]. Serre's GAGA (1956), which carries results between algebraic and analytic geometry in several complex variables, uses his own FAC (1955) [122, 123].

From K-theory to global analysis: the 1968 proof of the Atiyah–Singer index theorem uses the topological K-theory of Atiyah and Hirzebruch (1959/61) [124, 125, 126, 127]. And from logic and computation to information theory: Shannon's analysis of switching circuits (1937/38) applies Boole's algebra (1847/1854) [98, 99, 100].

Geometry, topology and analysis

From geometry to algebra: Omar Khayyam's geometric solution of cubic equations (about 1070–79) uses Euclid's Proposition VI.13 and Apollonius's Conics II.12 to intersect conic sections [24, 25, 26, 27]. From geometry to topology: Perelman's proof of the Poincaré conjecture (2002–03) uses Hamilton's Ricci flow of 1982 [28, 29, 30, 31].

From geometry to unifying structures: Hilbert's consistency proof for his axioms of geometry (1899) builds a model in the real numbers, and in that sense builds on Descartes' coordinates of 1637 (note 1) [32, 9, 33]. From geometry to Lie groups the crossing is one of influence only: Klein's ideas of 1871–72 shaped Lie's work on continuous groups of 1869–74 [34, 35].

From analysis to geometry: Archimedes' bounds on π (about 240 BCE) use the method of exhaustion of Eudoxus (about 370 BCE) [65, 66, 67]. From analysis to number theory: Hadamard and de la Vallée Poussin proved the prime number theorem (1896) with Riemann's complex zeta function and explicit formula of 1859 [62, 63, 64].

From analysis to probability and statistics: Kolmogorov's axioms (1933) rest, by his own account, on Lebesgue's measure and integral of 1902 [3, 68]. From analysis to numerical mathematics: Kantorovich's theorem of 1948, that Newton's method converges in Banach spaces, uses the theory of Banach spaces built by Riesz, Fischer and Banach (1920–32) [69, 70]. From integral equations to analysis, in turn: the work of Riesz and Fischer (1907) and of Banach (1920–32) uses the integral equations of Volterra (1896–97), Fredholm (1900–03) and Hilbert (1904–10) [154, 155].

From topology to combinatorics and graph theory: Kempe's attempted proof of the four-colour theorem (1879) and Heawood's five-colour proof (1890) use Euler's formula $V - E + F = 2$, through its consequence $E \leq 3V - 6$ [71, 72]. From combinatorics and graph theory to number theory: Green and Tao's theorem on

arithmetic progressions of primes (2004) uses Szemerédi's theorem (1975) as a black box [82, 83, 84].

From topology to game theory and economics: Nash's second existence proof for equilibria (1951) uses Brouwer's fixed-point theorem of 1909–12 [76, 77]. From game theory and economics to operations research: von Neumann's duality theory of linear programming (1947) uses the minimax theorem and the equivalence of linear programmes and games (MSC 91) [209, 210].

From complex analysis to probability and statistics: Schramm's stochastic Loewner evolution (1999/2000) uses Löwner's equation of 1923 to describe the conjectured scaling limit of loop-erased random walk and percolation; the crossing is stated, but what it reached was a conjecture [130, 131]. From complex analysis to control theory: Nyquist's stability criterion (1932) uses residues and the argument principle [132, 133, 134].

Potential theory crosses four times. Riemann's work of 1851 and 1857, on what later became unifying structures, uses the Dirichlet principle (1840s–1851) [135, 136]. The same principle is used in Riemann's dissertation and his mapping theorem (1851), in complex analysis [135, 137, 138]. Osgood's rigorous proof of the mapping theorem (1900) uses Green's function (Green, 1828) [138, 139, 140, 141]. The fourth crossing, to geophysics, is told below.

From special functions to complex analysis: the final step of de Branges's proof of the Bieberbach conjecture (1984/85) uses the Askey–Gasper inequality of 1976 [145, 146, 147]. From series and summability to abstract harmonic analysis the crossing is a new proof: Gelfand's theory of Banach algebras (1941) gave a new proof of Wiener's Tauberian theorem of 1932 [148, 149, 150]. From abstract harmonic analysis to Lie groups, a conjecture proved: Gleason, Montgomery and Zippin (1952) solved Hilbert's fifth problem [151, 152, 153].

From operator theory to linear algebra the crossing is a problem posed. Marcus, Spielman and Srivastava (2013) solved the Kadison–Singer problem of 1959 with a theorem on interlacing polynomials [158, 159, 160]. From the calculus of variations and optimisation to control theory: the control-theory chain describes Pontryagin's maximum principle (Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko, 1956) as a generalisation of the calculus of variations, though the calculus-of-variations chain gives no arrow (note 2) [161, 162, 163].

From global analysis to topology: Smale's proof of the Poincaré conjecture in dimension 5 and higher (1961) uses Morse theory (1925) [164, 165]. The same Morse theory gives Bott's proof of periodicity (1956–59), the second crossing from global analysis to K-theory [166, 167, 168, 169].

Into computation

From number theory to computer science, two crossings use Fermat's work. The RSA cryptosystem (1977) uses Euler's theorem, a^φ(n) ≡ 1 (mod n) [17, 18, 19, 20]. The AKS primality test (2002) uses a polynomial form of Fermat's little theorem [21, 22, 23].

From algebra to computer science: Babai's quasipolynomial algorithm for graph isomorphism (2015/17) uses group theory (MSC 20) throughout [57, 58, 59]. From probability and statistics to computer science: Blumer, Ehrenfeucht, Haussler and

Warmuth (1989) characterised learnability with the Vapnik–Chervonenkis theorem on uniform convergence [78, 79, 80, 81]. From operations research to computer science: the shortest-path algorithm of Duan and colleagues (2025) uses the Bellman–Ford method (MSC 90) as a subroutine (note 3) [206, 207, 208].

Into physics and the Earth

Geometry crosses into relativity twice. Einstein's field equations (1915) use the tensor calculus of Ricci and Levi-Civita, which Grossmann brought to him [36, 37, 38, 39]. Schoen and Yau's positive mass theorem (1979) has minimal surfaces as its background; the relativity chain gives no formal arrow (note 4) [40, 41, 42]. From geometry to geophysics: Eratosthenes' measurement of the Earth (about 230 BC) uses Greek geometry [43, 44].

From topology to relativity: Kehle and Unger's construction of collapse to extremal black holes (2022) is "modulated by the Borsuk–Ulam theorem" (note 3) [73, 74, 75].

Differential equations and dynamics cross five times. Helmholtz's vortex theorems (1858) use Euler's equations for an ideal fluid [85, 86, 87]. The non-uniqueness results of Buckmaster and Vicol (2019) and of Albritton, Brué and Colombo (2022) build on Leray's weak solutions of 1934 (note 1) [88, 89, 90, 91]. The invisibility counterexample of Greenleaf, Lassas and Uhlmann (2003) in optics builds on Calderón's inverse problem (MSC 35R30) (note 1) [92, 93]. Zermelo's objection to Boltzmann (1896) in statistical mechanics uses Poincaré's recurrence theorem of 1890 [94, 95]. And Kelvin's age of the Earth (1862/63) uses the heat equation [96, 97].

Two more crossings reach geophysics. Gauss, with Weber, used Laplace's equation in his geomagnetic work of 1832 and his general theory of 1839 [142, 143, 144]. Herglotz (1907) used Abel integral equations to recover seismic velocities inside the Earth [156, 157].

Mechanics receives and gives. Noether's theorem (1918) uses Lie groups of symmetries [128, 129]. Floer's homology (1987–89), in global analysis, uses the compactness of Gromov's pseudo-holomorphic curves of 1985, from the paper that also gave the non-squeezing theorem [170, 171, 172, 173]. Arnold's geometric view of fluids (1966) builds on Euler's rigid body, treating an ideal fluid as a rigid body on an infinite-dimensional group (note 1) [174, 175].

From fluid mechanics to astronomy: Jeans's instability criterion (1902) uses linearised fluid mechanics (MSC 76) [176, 177]. From optics and electromagnetism to mechanics: Hamilton's dynamics (1834–35) uses his optics of 1826–32 [178, 179, 180]. Lorentz's electron theory (1892–1904), on the way to relativity, has Maxwell's electrodynamics as its background (note 4) [181, 182].

From statistical mechanics to quantum theory: Planck's radiation law (1900) uses Boltzmann's $S = k \log W$ [183, 184]. From statistical mechanics to astronomy: Fowler's theory of white dwarfs (1926) uses the Fermi–Dirac statistics of the same year [185, 186].

From relativity to quantum theory: the Dirac equation (1928) uses special relativity (Einstein, 1905) [187, 188]. From relativity to astronomy, twice. The solutions of Schwarzschild (1916, exterior and interior), of Tolman, Oppenheimer and Volkoff (1939) and of Kerr (1963) use Einstein's field equations of 1915 [189, 190, 191, 192,

193, 194, 195, 196, 197]. Hawking radiation (1974–75) uses Bekenstein's black-hole entropy of 1972 [198, 199, 200].

The last crossing goes from astronomy to geophysics: in 1687 Newton deduced the oblate shape of the Earth from his law of gravitation [201, 202, 203, 204, 205].

Notes

1. In six crossings the dependency is inferred from the content rather than stated by a source: Descartes to Hilbert's consistency proof, *Moderne Algebra* to Zariski and Weil, Leray to the fluid non-uniqueness results, Calderón's problem to the invisibility counterexample, complete lattices to Birkhoff, and Euler's rigid body to Arnold.
2. Pontryagin's maximum principle stays as the control-theory chain states it, a generalisation, although the calculus-of-variations chain gives no arrow.
3. Two crossings name a result only through its classification: Borsuk–Ulam (MSC 55) has no link of its own in the topology chain, and Bellman–Ford (MSC 90) none in the operations-research chain. The Kehle–Unger abstract confirms the use of Borsuk–Ulam [73].
4. Two crossings are background only, since the receiving chain gives no formal arrow: minimal surfaces to Schoen–Yau, and Maxwell's electrodynamics to Lorentz.
5. In the map of crossings, one arrow is drawn per pair of fields (two where a pair has both a proof-use crossing and an inferred one), and only between neighbouring boxes, so it shows 49 of the 63 pairs. A solid arrow means a proof use (a conjecture proved counts); a dashed arrow means an inferred dependency, influence only, background, a problem posed, a new proof, a generalisation or a codification. Fourteen pairs appear in the list only: analysis → geometry, geometry → algebra, geometry → Lie groups, potential theory → unifying structures, complex analysis → probability, algebraic geometry → number theory, algebraic geometry → K-theory, algebra → computer science, mechanics → global analysis, differential equations → optics, differential equations → geophysics, integral equations → geophysics, optics → relativity and topology → relativity ($63 - 14 = 49$). The Leray crossing has its own dashed line beside the solid one for Helmholtz, and the minimal-surfaces crossing its own dashed line beside the solid Ricci-calculus one.
6. Two crossings were removed when the field chains were re-verified: linear algebra → associative rings and algebras (the Amitsur–Levitzki link no longer names the linear-algebra chain), and Euler's ideal-fluid equations → d'Alembert (1752), whose link now gives no arrow.
7. Fields with no crossing are left out: associative rings and algebras, non-associative algebras, category theory and homological algebra, approximation theory, integral transforms, convex and discrete geometry, elasticity and solids, thermodynamics, mathematical biology, mathematics education, the general section, mathematical modelling and the history of mathematics.
8. Machine-checked proofs are not counted as crossings, because they check existing proofs rather than use their results. They include the independence of the continuum hypothesis in Lean (2019–21), the odd order theorem in Coq (2012–13), Carleson's theorem in Lean (2025), Fermat's Last Theorem in Lean (September 2026), Flyspeck in HOL Light and Isabelle (2014), the dimension-8 sphere packing in Lean (2026), and the Lean checks in ∞-category theory, on

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Sources and method

Every link in this book rests on at least one opened source of standing. All 633 quick checks were recomputed in Python. Each chapter lists its own sources at its end, numbered as they are cited in the text.

The standard for a link

- **One good source is enough.** It can be a scholarly history, an encyclopaedia of mathematics, the original paper, or a university text. Wikipedia never counts as the only source.
- **Dependencies are stated or marked as inferred.** Where a source states that a proof uses an earlier result, the text says so plainly. Where the dependency follows from the content but no source states it, the text says “builds on” or “uses” and the chapter notes record that.
- **Single-source claims are attributed.** A claim that only one source makes is written as “according to” that source.
- **Independent results are named.** Parallel discoveries and rediscoveries are described as such, with no step drawn between them.
- **Conflicts are recorded.** Where sources disagree on a date, an attribution or a dependency, the chapter notes say so and say which source the text follows.
- **Preprints are marked.** A result with no journal version is called a preprint, with the date on which that was checked.

How the text was checked

An earlier draft of this book held its material in tables. A random sample of those tables found errors in about one row in ten in the first 27 fields and in 1–7% of rows in the rest. For this edition every link was read again for its person, its date, its statement and its dependency. A doubtful row was checked against a source and corrected where needed. That pass corrected eighteen statements of fact: dates (Markov’s count of letters in *Eugene Onegin* is from 1913; the disproof of Bernstein’s conjecture appeared in 1985), attributions (the fundamental theorem for K_1 of Laurent polynomials is due to Bass, Heller and Swan, 1964), and precise statements of theorems.

The draft also held 144 claims that rested on a single or a weak source. Each got one targeted search. 54 were confirmed by a further source, 79 are now attributed to the one source that states them, and 11 were left out.

Every quick check — a date difference, a formula, a bound or a numerical example — was recomputed in Python with sympy, numpy, mpmath or scipy. All 633 checks hold. Fourteen values were corrected or restated. Some were rounding ($\Gamma(\tfrac{1}{2})$ is exact in the Mellin example). Some depended on a random draw, and the text now gives the recomputed value. One was a span of years: the turbo-code patent ran 21 years, not 22.

The main kinds of source

- MacTutor History of Mathematics (St Andrews), for biographies and history topics: MacTutor, Lebesgue.
- Encyclopedia of Mathematics, for definitions and statements of theorems: EoM, Newton method.
- Stanford Encyclopedia of Philosophy, for foundations, logic and set theory: SEP, Russell’s paradox.
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