The Vacuum.Net Theory: Foundations

Ninth Edition — Leiden, 27 October 2026

J. Konstapel, Constable Research, Leiden

© 2026 J. Konstapel, Constable Research, Leiden. All rights reserved. Quotation with attribution is permitted; reproduction of the whole requires written permission.

Version 9.2. Companion: *The Vacuum.Net Theory: Physics*, Ninth Edition.

This document contains the axioms of the Vacuum.Net and everything that follows from them by proof. It makes no physical claim. Correspondences with physics, conversion units, identifications and predictions are in the companion document. A reader can check every statement here with pencil, paper and a short program, without accepting any physical reading first.

Part 0 — The Chain in One View

One strand, one rule, one register. Each step names the theorem where it is stated.

1. **One strand.** There is one strand. A distinction is a return of the strand to a place it has been (N1).
2. **The trinity.** Everything rests on the trinity: the three values −1, 0, +1, whose sum is zero. Written as the digits of balanced ternary, they add by the digit rule: a sum keeps its residue mod 3 at its own scale and passes the rest on as a carry to the next scale (T1).
3. **The window.** The shortest return that goes round takes three steps, so a window is three trits. A window closes when its sum vanishes mod 3; 9 of 27 close, in two kinds (T2–T6).
4. **Three elementary closures.** Every closed trit word breaks down completely into standings (0), meetings (x, −x) and rings (x, x, x) (T7). A word that closes everywhere repeats one closed window; the rings that close everywhere are the fixed points of the rewrite (T8–T10). Two different closed windows cannot be joined without tension (T11).
5. **The address.** Every integer has exactly one balanced-ternary word (T12). Open words, read from a start, number 3^n; closed configurations, read without a start, number g(n), the necklace count (T13). In the Eisenstein integers the same digits do not give every element a finite word (T14).
6. **Two ladders.** Merging meetings doubles, merging rings triples (T15). The ladders ×2 and ×3 never meet exactly; their best near-meetings are the closure depths 1, 2, 5, 12, 41, 53, 306, … and the comma (T16–T18).
7. **The first knot.** In three-dimensional space, three crossings are the least that cannot be undone; the trefoil is chiral. A rotor returns after 4π; the Hopf charge is conserved without Noether (T19–T22).
8. **Coupling.** The stiffness of the medium is a state function of its density (T23). For a free path the lever is −(1 + P)/P (T24).

Steps 1–6 use the axioms alone, with the hypotheses stated in T1 and T15. Steps 7 and 8 add stated hypotheses (an embedding in space; unit-quaternion orientation states; field maps $S^3 \to S^2$; an explicit energy density; a free random path). The companion document continues the chain with motion (Dirac, Rowlands), field (Maxwell), measure (Larson) and numbers.

Part I — Definitions and Axioms

I.1 Definitions

- An ordered index set I: the course of the strand.
- A set P of places.
- The strand: a map $\sigma: I \to P$.
- A loop: a pair $i < j$ in I with $\sigma(i) = \sigma(j)$. The strand returns to a place it has been.
- A node: a place visited at least twice. (The word *knot* is kept for knots in the sense of knot theory, Part VII.)
- A passage through a node: an index i with $\sigma(i)$ at that node. The first passage is the least such index.

I.2 Axioms

The theory rests on five axioms. Everything else is derived, corresponded, identified, predicted, or declared open. The statuses are defined in Part X.

N1 (Strand). There is exactly one strand. Every distinction is a loop.

N2 (Closure). A loop persists if and only if it lies in a closed window. What does not close passes on as tension. Closure is the selection rule; there is no external selector.

N3 (Memory). At every node the passages are ordered. The first passage fixes the direction; later passages are taken relative to it. The net carries its history as its shape.

N4 (Context). The response of the medium is a state function, not a universal constant. Universality is the empirical limiting case $g(\chi) = g_0$ when the state variable χ does not vary across the sample.

N5 (Scale). Merged into blocks, the coarse strand obeys N1 to N4 again. What does not close at one scale is the material of the next.

The ground: the trinity. Beneath the axioms lies one ground: the trinity, three values -1, 0, $+1$ whose sum is zero. The trits are not assigned to passages from outside; they are the digits of balanced ternary (III.2), and their arithmetic is the arithmetic of those digits (T1). The window used in N2 is fixed in Part IV (T2): a window is three consecutive trits, and it closes when it leaves no residue at its own scale.

Part II — The Ground of Distinction

The axioms rest on one act: a distinction. This part states where that act comes from.

II.1 The mark and its re-entry

Spencer-Brown's *Laws of Form* (1969) begins with one instruction: draw a distinction. A mark splits a space into an inside and an outside. Two laws govern marks. Calling twice is calling once. Crossing twice is not crossing.

Re-entry is the mark that contains itself. It cannot settle on marked or unmarked. It alternates: marked, unmarked, marked. Kauffman reads this alternation as the iterant $[+1, -1]$ together with the time shift η ($\eta^2 = 1$, $[a, b]\eta = \eta[b, a]$), and shows that $i = [+1, -1]\eta$ satisfies $i^2 = -1$. The imaginary unit is the oscillation of a self-referring distinction.

N1 states the same act in the language of the strand: every distinction is the strand returning to itself. A loop is a mark the strand makes on itself.

II.2 The eigenform as closure

Kauffman calls a form that reproduces itself under an operation an eigenform: $O(A) = A$. An eigenform is a fixed point of a transformation. N2 says that only configurations that close persist. On the register, a ring closes in every window exactly when it is a fixed point of the rewrite R (T10). N2 thus takes the form of the eigenform condition for R. The open question IX.2 asks for the rewrite dynamics of the whole net that is its own fixed point.

II.3 Status

The mark, re-entry and the eigenform are the ground of N1 and N2. They are declared, not derived. The reading of IX.2 as the eigenform question is a correspondence.

Part III — Direction and the Trit

III.1 Three values

T1 (The trinity as register). *Ground.* The trinity: the three values -1, 0, $+1$, written T, 0, 1, whose sum is zero. Read as the digits of balanced ternary, they add by the digit rule: for a sum s of trits, the digit is the balanced residue $d \in \{-1, 0, +1\}$ with $d \equiv s \pmod 3$, and the carry $(s - d)/3$ passes to the next position, one scale step ($\times 3$) higher. *Statement.* (i) A sum of trits leaves no residue at its own scale if and only if $s \equiv 0 \pmod 3$. (ii) Whatever it does not keep passes on, whole, as carry to the next scale. (iii) The residues form $\mathbb{Z}_3$, the smallest cyclic group in which a value differs from its reverse. Status: proved (arithmetic of balanced ternary), given the ground.

Proof. Every integer s has exactly one balanced residue d ∈ {−1, 0, +1} with s − d divisible by 3 (T12, existence and uniqueness of the last digit). The digit of s is d; the carry is (s − d)/3. The residue is 0 exactly when 3 divides s. The residues add modulo 3, which is $\mathbb{Z}_3$; in $\mathbb{Z}_1$ and $\mathbb{Z}_2$ every element equals its reverse (1 = −1 in $\mathbb{Z}_2$), in $\mathbb{Z}_3$ it does not (−1 = 2 ≠ 1). ∎

The trits are therefore not directions that compose but digits that add. Read as directions, “the same as the first” would be the neutral element and “the reverse” its own inverse, a structure of two values; that reading is not used. Read as digits of the trinity, the rule of N2 and N5 is arithmetic: what closes at a scale leaves digit 0 there, and what does not close is either kept as a nonzero digit (tension, N2) or carried to the next scale (N5). The trit rule below reads off, for each later passage, which digit it contributes.

Definition (trit rule). Let i be a later passage through a node, not the last index of the strand, and let f be the first passage through σ(i). Then t(i) = +1 if σ(i + 1) = σ(f + 1): it leaves the node the way the first passage left it; t(i) = −1 if σ(i + 1) = σ(f − 1): it leaves the way the first passage came in; t(i) = 0 otherwise. The rule uses only σ. It is mechanical.

Remark (algebraic home). With ω a primitive cube root of unity and β = 1 − ω, of norm 3, the ring Z[ω]/(β) is the field with three elements. Since ω ≡ 1 mod β, the class of a + bω is a + b mod 3, and its natural representatives are −1, 0, +1. The trit is the residue modulo the prime above 3 in the Eisenstein integers. T14 shows how far this carries.

Remark (radix economy). The cost of writing N in radix r is r · ⌈log_r(N + 1)⌉, which per unit of ln N tends to r / ln r: 2.885 for r = 2, 2.731 for r = 3, 2.885 for r = 4, with the continuous optimum at r = e = 2.718. Three is the integer radix closest to the optimum. This is Leibniz’s principle of maximal richness from minimal means (*Discours de métaphysique*, 1686, §5–6), in Chaitin’s measurable form: a theory is a program shorter than the data it produces. It agrees with T1 but does not prove it; T1 rests on direction, not on cost.

III.2 The balanced notation

Base three has two notations. Unbalanced ternary uses {0, 1, 2}. Balanced ternary uses {−1, 0, 1}, written here as {T, 0, 1}. The balanced form is the notation of the strand: T is counterwinding, 0 is rest, 1 is winding. It closes without external conventions. Four facts, each verifiable by direct computation:

1. *No minus sign.* The sign of a number is the sign of its leading nonzero trit. Negation is trit-wise mirroring: swap 1 ↔ T. Example: 8 = 10T (9 − 1); −8 = T01 (−9 + 1).
2. *Closed single-trit multiplication, no carry.* 1×1 = 1, 1×T = T, T×T = 1, anything ×0 = 0. The addition table has two carries in nine entries. Unbalanced ternary has three carries in nine addition entries and a carry in 2 × 2 = 11.
3. *Truncation equals rounding* (Knuth). Cut a balanced-ternary expansion after position m. The discarded tail is at most $\Sigma_{i>m} 3^{-i} = 3^{-m}/2$ in absolute value — half the last kept unit. Cutting is automatically rounding to nearest.

4. *Shift equals scale step.* Shifting the trit string one place left multiplies by 3; one place right divides by 3. Same string, other depth.

Decimal and binary in their standard signed form need two external conventions: a minus sign and a rounding rule. Balanced ternary needs neither. Among positional notations with radix 3 and three integer digits it is the only one that needs neither: the tail bound $\max|d| \cdot 3^{-m}/2 \le 3^{-m}/2$ forces $|d| \le 1$, and three digits forming a complete residue system mod 3 within $\{-1, 0, 1\}$ are exactly $\{T, 0, 1\}$. (Radix -3 with the same digits is the same notation with odd positions mirrored.)

Part IV — Windows and Closure

IV.1 The window

T2 (Window of three). A run of equal nonzero trits leaves no residue first when it has length three. A window is therefore three consecutive trits; it closes when $x + y + z \equiv 0 \pmod 3$, that is, when its digit is 0. Status: proved (on T1).

Proof. A run of k equal trits $x \neq 0$ has sum kx, whose residue is 0 first at $k = 3$ (1 and 2 are not divisible by 3; 3 is). By T1, a run leaves no residue exactly when its sum is 0 mod 3. ∎

A closed run of trits returns to its start, so it is itself a loop one scale up. This is N5 at work.

T3 (Nine of twenty-seven). Exactly 9 of the 27 windows close. They form a subgroup of $\mathbb{Z}_3^3$ of index 3. Status: conditionally proved; computed by enumeration.

Proof. The sum map $\mathbb{Z}_3^3 \to \mathbb{Z}_3$ is a surjective homomorphism. The closing windows are its kernel, of order $27/3 = 9$. ∎

T4 (Two kinds of closure). A window closes if and only if its three trits are all equal or all different. There are 3 homogeneous and 6 heterogeneous closing windows; the 18 open windows all have exactly two equal trits. Status: conditionally proved.

Proof. All equal to a gives $3a \equiv 0$. All different gives $1 + 0 + (-1) = 0$. Otherwise exactly two are equal: the sum is $2a + b$ with $b \neq a$; $2a + b \equiv 0$ would give $b \equiv -2a \equiv a$, a contradiction. The open windows: 3 values for the pair, 2 for the other trit, 3 positions: 18. ∎

This is the rule of Fox three-colouring at a crossing, used in T19. The register and the first knot obey one rule.

T5 (The trit cube). Read $\{T, 0, 1\}^3$ as a cube. The number of cells with k nonzero coordinates is $C(3, k) \cdot 2^k$: 1, 6, 12, 8. The closing cells are the centre, the six cells with two nonzero coordinates of opposite sign, and (1, 1, 1) and (T, T, T): $1 + 0 + 6 + 2 = 9$. Status: conditionally proved; computed.

Proof. One nonzero coordinate gives ±1. Two give ±2 when the signs agree and 0 when they differ. Three give $\pm 3 \equiv 0$ when all signs agree, and ±1 otherwise. ∎

T6 (Closure probability). Let the three trits be independent with law (p_0, p_+, p_-). Put $\hat{p}(z) = p_0 + p_+ z + p_- z^2$ and $\omega = e^{2\pi i/3}$. A window closes with probability $(1/3) \cdot [1 + \hat{p}(\omega)^3 + \hat{p}(\omega^2)^3]$. The uniform law gives 1/3. Status: conditionally proved; checked on random laws.

Proof. The indicator of $s \equiv 0 \pmod 3$ is $(1/3) \cdot \Sigma_k \omega^{ks}$. Take expectations over independent entries. ∎

IV.2 Elementary closures

T7 (Elementary closures). Every closed trit word reduces to nothing by removing, one at a time, a standing 0 (length 1), a meeting x, −x (length 2) or a ring x, x, x (length 3). No open word does. Among closures whose steps are all nonzero, a meeting is the shortest with winding number 0 and a ring the shortest with winding number ±1. Status: conditionally proved; computed for all 265,720 closed words of length 1 to 12.

Proof. Each removal takes away a piece with sum 0 mod 3, so it keeps the sum of the word mod 3; only closed words can reduce to nothing. Conversely, let w be closed. Remove every 0. If the remaining word has neighbours x, −x, remove them. If it has none, all its entries are equal, because two different nonzero trits are opposite; its length L then satisfies $L \cdot x \equiv 0$, so 3 divides L, and it splits into rings. Each step shortens the word and keeps it closed; by induction it reduces to nothing. For the second part, lift the nonzero steps ±1 to the integers. A walk returns on $\mathbb{Z}_3$ if and only if its displacement is 3w for a whole number w, and the displacement is at most the length. So $|w| = 1$ needs at least three steps, and x, x, x reaches it; $w = 0$ with nonzero steps needs an even length, at least 2, and x, −x reaches it. ∎

Apart from standing still there are two ways to close: to meet and to go round. Their lengths are 2 and 3. Part VI builds the two ladders from them.

Corollary T7.1 (What a closure carries). A standing and a meeting close with carry 0: nothing passes to the next scale. A ring closes with carry ±1: 1 + 1 + 1 = 10 and T + T + T = T0 in balanced ternary. A ring therefore closes at its own scale and becomes one passage of the next — the rule of N5, as arithmetic. An open window keeps a nonzero digit (tension, N2) and may also carry: 1 + 1 = 1T. *Proof.* Apply the digit rule of T1 to the sums 0, ±3 and ±2. ∎

IV.3 Words and rings that close everywhere

T8 (Propagation; nine words). Let $s_k = t_k + t_{k+1} + t_{k+2}$ be the sum of window k of a word. Then $s_{k+1} \equiv s_k + t_{k+3} - t_k \pmod 3$. A word of length $n \geq 3$ closes in every window if and only if its first window closes and it has period 3. For every $n \geq 3$ there are exactly 9 such words. Status: conditionally proved; computed for n = 3 to 9.

Proof. $s_{k+1} = t_{k+1} + t_{k+2} + t_{k+3} = s_k - t_k + t_{k+3}$. If every window closes, $s_{k+1} \equiv s_k \equiv 0$, so $t_{k+3} \equiv t_k$, hence $t_{k+3} = t_k$, since distinct trits are distinct mod 3. Conversely, if the first window closes and the word has period 3, every window has the same sum 0. A word of period 3 is fixed by its first three trits, which form a closed window: 9 choices by T3. ∎

A path without tension is one of nine: a constant run, or one order of (−1, 0, +1) repeated.

T9 (A ring that closes everywhere). A ring is a closed strand of n ≥ 3 trits, read cyclically. If every window of a ring closes, then $t_{i+3} = t_i$ for all i. Counted with labelled positions there are 9 such rings when 3 divides n, and 3 otherwise. Counted as rotation classes — closed configurations without a starting point (T13) — there are 5 when 3 divides n (the three constant rings and two classes of period 3) and 3 otherwise. Status: conditionally proved; computed for n = 3 to 13 (labelled) and n = 3, 6, 9 (classes).

Proof. By T8, $t_{i+3} = t_i$. So 3 is a period, and so is n; hence gcd(3, n) is a period. If 3 divides n, choose t_0 and t_1 freely and put $t_2 = -(t_0 + t_1)$; every window is a cyclic shift of (t_0, t_1, t_2) and closes: 9 rings. Otherwise the ring is constant, t = a, and 3a ≡ 0 always: 3 rings. Rotation acts on the 9 rings with three fixed rings (the constants) and two orbits of length 3 (the six heterogeneous windows, which are the two cyclic orders of (−1, 0, +1) in their three rotations): 3 + 2 = 5 classes. ∎

T10 (The rewrite of the register). Let R act on a ring by $R(t)_i = -(t_{i-1} + t_{i+1})$. A ring is a fixed point of R if and only if every window closes. Status: conditionally proved.

Proof. $R(t)_i = t_i$ means $t_{i-1} + t_i + t_{i+1} \equiv 0$. ∎

Since 2 ≡ −1 (mod 3), $R(t)_i = 2^{-1}(t_{i-1} + t_{i+1})$: each trit becomes the mean of its neighbours in $\mathbb{Z}_3$. The fixed points are the eigenforms of the register (II.2). They are fixed, not attracting: for n = 3 to 10, the rings that are not fixed end in cycles, except 24 rings for n = 4 and 24 for n = 8, which end in a fixed point. Computed.

IV.4 Seams

T11 (Seam rule). A seam joins a closed window A to a closed window B in the word $A_1A_2A_3B_1B_2B_3$. The number of open windows at the seam is $[A_1 \neq B_1] + [A_3 \neq B_3]$. The seam is closed if and only if B = A. Over all 81 ordered pairs: 9 closed seams, 36 with one open window, 36 with two. Status: conditionally proved; computed.

Proof. The windows across the joint are $W_1 = (A_2, A_3, B_1)$ and $W_2 = (A_3, B_1, B_2)$. Since A closes, $A_2 + A_3 \equiv -A_1$, so the sum of W_1 is $\equiv B_1 - A_1$: W_1 closes if and only if $B_1 = A_1$. Since B closes, $B_1 + B_2 \equiv -B_3$, so the sum of W_2 is $\equiv A_3 - B_3$: W_2 closes if and only if $B_3 = A_3$. A closed window is fixed by its first and last trit, because its middle is ≡ −(first + last). So the seam is closed if and only if B = A. For fixed A, of the 9 windows B: 1 gives no open window, 4 give one, 4 give two. ∎

A window continued in its own order gives no tension. A seam between two different closed windows always carries tension.

Part V — Addresses

V.1 Unique address in the integers

T12 (Unique address). Every nonzero integer has exactly one finite balanced-ternary representation $\Sigma\, d_k 3^k$ with $d_k \in \{T, 0, 1\}$ and nonzero leading trit; 0 is represented by the empty string (written 0). Words of k digits give exactly the integers from $-(3^k - 1)/2$ to $(3^k - 1)/2$. Status: proved (pure arithmetic).

Proof. Existence. Let $|N| \le (3^k - 1)/2$. Then $M = N + (3^k - 1)/2$ lies in $[0, 3^k - 1]$ and has a standard ternary expansion $M = \Sigma_{i<k}\, c_i 3^i$ with $c_i \in \{0, 1, 2\}$. Since $(3^k - 1)/2 = \Sigma_{i<k}\, 3^i$, $N = \Sigma_{i<k}\, (c_i - 1)3^i$ with $c_i - 1 \in \{T, 0, 1\}$. Delete leading zeros. Uniqueness. Suppose two distinct finite trit strings represent the same integer. Pad the shorter with leading zeros. Their trit-wise difference gives $\Sigma\, d_i 3^i = 0$ with all $d_i \in \{-2, \ldots, 2\}$, not all zero. Let k be the largest index with $d_k \neq 0$. Then $|d_k| 3^k \ge 3^k$, but $|\Sigma_{i<k}\, d_i 3^i| \le 2\, \Sigma_{i<k}\, 3^i = 3^k - 1 < 3^k$. The sum cannot vanish. ∎

Corollary T12.1 (Mirror charge). The address of −Q is the address of Q, mirrored trit-wise ($1 \leftrightarrow T$). The sign of a charge is a property of the string, not a label attached from outside.

Corollary T12.2 (Self-centering depth). By fact (c) of III.2, the value read at finite depth m is automatically a nearest value at that depth (at an exact midpoint, where the expansion ends in 111… or TTT…, both neighbours are nearest and both expansions exist). Description at a given depth needs no external rounding convention.

Corollary T12.3 (Capacity and depth). The largest value expressible with n digits is the all-ones string: $(3^n - 1)/2 = 1, 4, 13, 40, 121, \ldots$ A number $n \ge 1$ has exactly $\lfloor \log_3(2n - 1) \rfloor + 1$ digits. *Proof of the digit count.* n has d digits if and only if $(3^{d-1} - 1)/2 < n \le (3^d - 1)/2$, that is $3^{d-1} < 2n + 1$ and $2n - 1 < 3^d$. Since 3^{d-1} and $2n + 1$ are odd, the first gives $3^{d-1} \le 2n - 1$. So $3^{d-1} \le 2n - 1 < 3^d$. ∎ On the ring ladder (T15 with a = 0) digit k stands at scale 3^k, so the number of digits measures depth in scales.

Precedent. Euler stated the uniqueness in T12 as an identity of formal Laurent series,

$$\prod_{k \ge 0} (x^{-3^k} + 1 + x^{3^k}) = \Sigma_{n \in \mathbb{Z}}\, x^n$$

(Andrews 2007, §1 and §5, Theorem 4). The notation runs on hardware: Brusentsov's Setun computer (Moscow, 1958) computed in balanced ternary. Current ternary-weight neural networks, including the 1.58-bit large-language-model line (Ma et al. 2024), use exactly the alphabet $\{-1, 0, 1\}$.

V.2 Open words and closed configurations

T13 (Two counts). (i) A path — the strand on an interval of I — has exactly one trit word, fixed by σ through the trit rule. Of the 3^n open words of length $n \ge 3$, 3^{n-1} close their first window and exactly 9 close everywhere (T8). (ii) A closed configuration of n

trits, read without a starting point, is a rotation class of words. The number of rotation classes is the necklace count

$$g(n) = (1/n) \cdot \Sigma_{d \mid n}\, \varphi(n/d) \cdot 3^{d} = 3, 6, 11, 24, 51, 130 \text{ for } n = 1, \ldots, 6.$$

Open words, read from a start, number 3^n; closed configurations number g(n). Status: conditionally proved; computed.

Proof. (i) A word of length n has 3^n choices; its first window closes in 9 of 27 cases and the other n − 3 trits are free: $9 \cdot 3^{n-3} = 3^{n-1}$. (ii) The cyclic group of order n acts on the 3^n words by rotation. A rotation by j fixes exactly $3^{\gcd(j,n)}$ words. By Burnside's lemma the number of orbits is $(1/n)\ \Sigma_j\ 3^{\gcd(j,n)}$, which, grouped by d = gcd(j, n), is the formula stated. ∎

When the depth of a state is counted with a starting point, each step deeper multiplies the count by three. When a closed configuration is counted without one, the count is g(n), which is close to $3^n/n$ for large n.

V.3 The limit of the address in the Eisenstein integers

T14 (Rest and turn in Z[ω]). Let β = 1 − ω, let d(z) ∈ {−1, 0, +1} be the balanced residue of z = a + bω, so d ≡ a + b (mod 3), and put T(z) = (z − d(z))/β. Then every orbit of T reaches either 0 or one of the two fixed points ±(1 + ω) = ∓ω², each a sixth root of unity: a 60° turn. An element has a finite trit word $z = d_0 + d_1\beta + d_2\beta^2 + \ldots$ if and only if its orbit reaches 0; otherwise its word ends in a repeated digit (−1 repeated for 1 + ω, +1 repeated for −(1 + ω)). Not every element has a finite word: 2 → 2 + ω → 1 + ω. The share of elements with |a|, |b| ≤ 40 whose orbit ends at a fixed point is 1,788 of 6,561 (27.3 %); for |a|, |b| ≤ 100 it is 16,546 of 40,401 (41.0 %). Status: conditionally proved; computed.

Proof. z − d(z) ≡ 0 mod β, so T(z) lies in Z[ω]. The norm drops: $N(T(z)) = N(z - d)/3 \le (\sqrt{N(z)} + 1)^2/3$, which is smaller than N(z) whenever N(z) ≥ 2. So every orbit enters the elements of norm 0 or 1: zero and the six units. By direct computation 0, ±1 and ±ω run to 0, while ±ω² = ∓(1 + ω) map to themselves: (1 + ω)(1 − ω) = 1 − ω² = 2 + ω, so T(1 + ω) = (1 + ω + 1)/β = 1 + ω. ∎

T14 is the boundary of T12. In ℤ the trits give every number one finite word. In Z[ω], with base 1 − ω, they give a finite word only to the elements that come to rest; the others end in a perpetual turn. Whether the share settles at a limit as the box grows is open (IX.3).

Part VI — Scale: Two Ladders

T15 (Ladder). *Hypothesis.* At each scale the strand is merged into blocks of equal size, each block one elementary closure of length 2 (a meeting) or 3 (a ring) (N5, T7). *Statement.* After a merges of meetings and b merges of rings, one coarse passage contains exactly $2^a \cdot 3^b$ passages of the finest scale. On the ring ladder (a = 0) each step multiplies by three. Status: conditionally proved.

Proof. At each scale every coarse passage is one block, and every block at that scale has the same size: 2 for a merge of meetings, 3 for a merge of rings. A coarse passage therefore holds the product of the block sizes below it. ∎ The equal size is needed: a ring merged into one passage and then joined to a single passage in a meeting gives 3 + 1 = 4 finest passages, not 2 · 3 = 6.

T16 (No exact meeting). $2^n \neq 3^k$ for all integers n, k other than n = k = 0. So $\log_2 3$ is irrational, and the fractional parts of $k \cdot \log_3 2$ ($k \geq 1$) are dense in [0, 1). Status: proved.

Proof. For n > 0, 2^n is even and 3^k is odd; for n = 0, $3^k = 1$ forces k = 0. An irrational step generates a dense set (Kronecker; Hardy and Wright, Theorem 439). ∎

Under the merging of T15, meetings and rings never return together: the ladders ×2 and ×3 never close onto each other.

Definition (closure depth). For $k \geq 1$ let $\|k \cdot \log_2 3\|$ be the distance from $k \cdot \log_2 3$ to the nearest integer n. A closure depth is a k with $\|k \cdot \log_2 3\| < \|j \cdot \log_2 3\|$ for every $1 \leq j < k$. It counts powers of 3.

T17 (The closure depths). The closure depths are the denominators of the convergents of $\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, \ldots]$. Up to 10^6 they are

1, 2, 5, 12, 41, 53, 306, 665, 15601, 31867, 79335, 111202, 190537.

Status: proved (Lagrange’s theorem on best approximations of the second kind; Khinchin, Chapter II, §6); computed for $k \leq 10^6$.

Proof. A best approximation of the second kind of θ is a fraction n/k with $|k\theta - n| < |j\theta - m|$ for all $0 < j < k$ and all m. By Lagrange’s theorem (Khinchin, Theorems 16 and 17) these are exactly the convergents p_m/q_m, with the single exception of $p_0/q_0 = a_0$ when $a_1 = 1$, as here. For $k \geq 2$ the definition of a closure depth is this condition, with n the nearest integer to kθ. k = 1 is a closure depth vacuously; it is the denominator of $p_1/q_1 = 2/1$. So the closure depths are $q_1, q_2, q_3, \ldots$ ∎

T18 (The comma). At depth 12 the two ladders miss by $3^{12}/2^{19} = 531441/524288 = 1.013643$, the Pythagorean comma (23.46 cents). The miss never vanishes (T16). Status: proved; computed. The residues at the closure depths:

Depth k	**n**	**Comparison**	**Residue $\lvert 2^n/3^k - 1\rvert$**	**Residue in cents, $1200 \cdot \lvert k \cdot \log_2 3 - n\rvert$**
1	2	$3^1 \sim 2^2$	33.3 %	498.045
2	3	$3^2 \sim 2^3$	11.1 %	203.910
5	8	$3^5 \sim 2^8$	5.35 %	90.225
12	19	$3^{12} \sim 2^{19}$	1.35 %	23.460
41	65	$3^{41} \sim 2^{65}$	1.15 %	19.845
53	84	$3^{53} \sim 2^{84}$	0.21 %	3.615
306	485	$3^{306} \sim 2^{485}$	0.10 %	1.770

Depth k	n	Comparison	Residue $\lvert 2^n/3^k - 1\rvert$	Residue in cents, $1200 \cdot \lvert k \cdot \log_2 3 - n\rvert$
665	1054	$3^{665} \sim 2^{1054}$	0.0044 %	0.0756
15601	24727	$3^{15601} \sim 2^{24727}$	0.0018 %	0.0315

Between the closure depths, nothing closes better. Each near-closure is finite: a cycle at depth 665 holds $3^{665} \approx 1.9 \times 10^{317}$ finest passages on the ring ladder.

Part VII — The First Knot

This part adds one hypothesis to the axioms. *Hypothesis (embedding).* The strand is embedded in three-dimensional space. Seen in a planar projection, a node where the strand passes itself is a crossing with three states: over (+1), under (−1), or none (0). Under this hypothesis the knots of knot theory become available.

T19 (First knot). *Hypotheses.* The embedding hypothesis; closed embedding (S^1 or fixed endpoints); crossings countable. *Statement.* The trefoil is the simplest configuration that cannot be undone, at three crossings. Status: conditionally proved.

Proof. Distinctness. Colour the arcs of a diagram with three colours such that at every crossing the three meeting arcs carry either one colour or all three. The crossingless unknot diagram has one arc and so admits only single-colour colourings; by the invariance below, no diagram of the unknot admits a colouring with more than one colour. The standard trefoil diagram admits a colouring using all three colours: assign each of its three arcs a different colour; at each crossing all three colours meet. Tricolourability is invariant under all three Reidemeister moves (checked case by case on the finitely many local pictures). Hence the trefoil is not the unknot. Minimality. Diagrams with 0, 1 or 2 crossings are finitely many up to planar isotopy: the only planar Gauss words are ∅, 11, 1122 and 1221 (the word 1212 violates Gauss's even-interlacing condition). In each of them some crossing is a kink removable by a Reidemeister I move, and repeating this reduces the diagram to the unknot. For fixed endpoints, the endpoints lie on the boundary of the ambient ball (or at infinity); closing the arc along the boundary identifies these arc classes with knot classes, so the closed case covers it. (Endpoints fixed at interior points would not suffice: every such arc can be undone.) So no nontrivial knot exists below three crossings. At three crossings, the only planar Gauss word without a kink is 123123, the trefoil shadow. Of its eight crossing choices, the two alternating ones are the right- and left-handed trefoils. Each of the six non-alternating ones has a bigon whose one edge passes over at both ends; a Reidemeister II move removes that bigon and leaves one crossing, hence the unknot. So every nontrivial three-crossing knot is a trefoil. ∎

The proof itself is ternary: three arcs, three colours, three crossings. The colouring rule is the window rule of T4.

T20 (Bracket of the first knot). A knot diagram changes without changing the knot by exactly three local moves (Reidemeister 1927). Kauffman's bracket is invariant under two of them and fixed by three rules: $\langle O \rangle = 1$; adding a free circle multiplies by $d = -A^2 - A^{-2}$; a crossing splits into its two smoothings, $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$. For the standard three-crossing diagram of the trefoil, $\langle K \rangle = A^{-7} - A^{-3} - A^5$. Normalised by the writhe $w = 3$, $f(K) = (-A^3)^{-w}\langle K \rangle$, and with $A = t^{-1/4}$, the Jones polynomial is $V(t) = t + t^3 - t^4$. Its mirror image gives $V(t) = t^{-1} + t^{-3} - t^{-4}$. Status: conditionally proved (state sum over the $2^3 = 8$ smoothings; computed).

Proof. Each of the 2^3 states chooses one smoothing at each crossing. A state with a A-smoothings, b B-smoothings and $|s|$ circles contributes $A^{a-b} \cdot d^{|s|-1}$. The eight states give the sum stated. Multiplying by $(-A^3)^{-3}$ and substituting $A = t^{-1/4}$ gives V. ∎

The unknot has $V = 1$. So the trefoil is not the unknot, and it differs from its mirror image: the first knot is chiral. The three-colouring of T19 and the bracket are linked exactly: the number of Fox three-colourings equals $3|V(e^{i\pi/3})|^2$ (Przytycki 1998; Lickorish and Millett 1986). For the trefoil $V(e^{i\pi/3}) = i\sqrt{3}$, which gives 9. The bracket also detects chirality, which the colouring does not.

T21 (Double cover). *Hypotheses.* Orientation states form the unit quaternions; a strand is attached (has ends or environment). *Statement.* A strand returns to its initial state after a rotation of 4π, not 2π. Status: conditionally proved.

Proof. Unit quaternions $q = a + bi + cj + dk$ with $i^2 = j^2 = k^2 = -1$ represent rotations by $v \to q\,v\,q^*$ on pure-imaginary quaternions v. Both q and $-q$ give the same rotation: the representation is a double cover. A continuous rotation by 2π carries q to $-q$; only 4π returns q to q. For a free frame the sign is unobservable. For an attached strand the configuration records the homotopy class of the rotation path in SO(3) relative to the attachment. $\pi_1(SO(3)) = \mathbb{Z}/2$: the 2π loop is the nontrivial class (its lift starting at q ends at $-q \neq q$), the 4π loop is trivial. So the 2π-rotated attached configuration cannot be deformed back to the initial one with the attachment fixed, while the 4π-rotated one can (Dirac belt trick). ∎

T22 (Charge conservation). *Hypotheses.* Field configurations map into the target space of closed windings; the strand is continuous (N1); no boundary is crossed. *Statement.* The Hopf charge $Q \in \pi_3(S^2) = \mathbb{Z}$ is conserved under all continuous rewrites. Status: conditionally proved.

Proof. Q is an integer-valued homotopy invariant. A field configuration is a map $S^3 \to S^2$ (space compactified by a fixed value at infinity, which is what "no boundary is crossed" secures), and a continuous rewrite is a homotopy of such maps. Since S^2 is simply connected, free and based homotopy classes agree: $[S^3, S^2] = \pi_3(S^2) = \mathbb{Z}$, the integer being the Hopf invariant (the linking number of the preimages of two regular values; for the Hopf map these are two fibres). A continuous deformation cannot change an integer continuously. Q can change only through discontinuity: cutting the strand, leaving the target space, or crossing a boundary. All three are excluded by hypothesis. No symmetry argument (Noether) is invoked. ∎ By T12.1, the addresses of Q and $-Q$ are trit-wise mirrors.

Part VIII — Coupling

T23 (Constitutive equation). *Hypotheses.* The energy density of the medium is $e(n) = (g_2/2)n^2 + (g_3/6)n^3$, with $g_3 \neq 0$. A reference density $\bar{n} \neq 0$ is chosen such that $\bar{g} \equiv g_2 + g_3\bar{n} \neq 0$. Scope: one explicit realization of N4. *Statement.* $g_eff = \bar{g}(1 + \gamma\delta)$, with $\gamma \equiv g_3\bar{n}/\bar{g}$ and $\delta \equiv (n - \bar{n})/\bar{n}$. Consequently $g_2 = \bar{g}(1 - \gamma)$. Status: conditionally proved.

Proof. Two derivatives: $g_eff = d^2e/dn^2 = g_2 + g_3 n$. Insert $n = \bar{n}(1 + \delta)$: $g_eff = (g_2 + g_3\bar{n}) + g_3\bar{n}\delta = \bar{g}(1 + \gamma\delta)$. From $\gamma = g_3\bar{n}/\bar{g}$ and $\bar{g} = g_2 + g_3\bar{n}$: $g_2 = \bar{g}(1 - \gamma)$. ∎

The stiffness of the medium is a state function of its density: N4 made explicit. By N4, γ is itself a state function; no fixed value is asserted. For $\bar{g} > 0$ the quadratic coefficient g_2 is negative when $\gamma > 1$ and positive when $\gamma < 1$.

Definition (lever of a count). For an integer-valued count V with mean $\bar{n}$ and cumulants $\kappa_2 > 0$, κ_3, the lever is $\gamma(V) = -\bar{n} \cdot \kappa_3/\kappa_2^2$. For a smooth closing energy whose fluctuations $x = n - \bar{n}$ have weight $\exp(-N \cdot f(x))$ near the minimum, with $f(x) = (a/2)x^2 + (b/6)x^3$, $a = e''(\bar{n}) > 0$ and $b = e'''(\bar{n})$, the lever equals $\bar{n} \cdot b/a = \bar{n} \cdot e'''(\bar{n})/e''(\bar{n})$, which is γ of T23, to leading order in $1/N$ (Laplace's method). The lever is not shift-invariant, so the count must be stated exactly.

T24 (Lever of a free path). *Hypotheses.* The strand is a free path: the simple random walk on $\mathbb{Z}^d$, $d \geq 3$, with return probability $P_d < 1$. A node counts the visits V to its place, the first included. *Statement.* V is geometric, $P(V = k) = (1 - P_d) \cdot P_d^{k-1}$, and its lever is $\gamma = -(1 + P_d)/P_d$. With $P_3 = 0.3405373$, $P_4 = 0.1932017$ and $P_5 = 0.1351786$ (Watson 1939; Montroll 1956; OEIS A086232, A086233), $\gamma = -3.9365$, -6.1759 and -8.3976. Status: conditionally proved; computed.

Proof. By the strong Markov property each visit is followed by another with probability P_d, independently. With $q = 1 - P_d$: $\bar{n} = 1/q$, $\kappa_2 = P_d/q^2$, $\kappa_3 = P_d(1 + P_d)/q^3$. Then $-\bar{n} \cdot \kappa_3/\kappa_2^2 = -(1 + P_d)/P_d$. ∎

Finite walks. A walk of finite length L counts fewer returns. For $d = 3$ and $L = 10^4$ the exact return probabilities, combined through the renewal equation, give a return probability within the walk of 0.337668; the exact distribution of V then gives the lever -3.9612. As L grows the lever tends to -3.9365. The lever depends on the freedom of the path, through P_d. That the lever of the free path equals γ of the medium is not proved here (IX.1).

Part IX — Open Questions and Position

IX.1–IX.5 Open questions

IX.1 Lever and coupling. Whether the lever of a free path (T24) is the coupling γ of the medium (T23), and with it the sign of g_2 at the reference state.

IX.2 Closure dynamics (Rosen). Not "which action does the net minimize" but "which rewrite dynamics is its own fixed point". On the register this is solved: the fixed points

of the rewrite are the rings that close everywhere (T9, T10), the eigenforms of II.2. Open for the whole net.

IX.3 The share in Z[ω]. Whether the share of elements of Z[ω] whose digit orbit ends in the turn (T14) settles at a limit as the box grows, and which elements end there.

IX.4 Equal blocks. Under which conditions a strand merges in blocks of equal size, so that T15 applies, and how mixed merging changes the ladder.

IX.5 The trit rule as digit. T1 settles the arithmetic of the trits from the trinity. Open is the physical reading of the trit rule: that the passage leaving a node the way the first passage left it contributes the digit +1, the reverse −1, and any other 0.

IX.6 Position

Wolfram and Gorard select rules by the criterion that known physics emerges; the ruliad is an external standard. Program length gives a neutral comparison (III.1, remark): richness produced per axiom. Five axioms here; the challenge to any rival is to produce the same richness from fewer. Verlinde moves the foundation one floor (holography, de Sitter) but keeps it external. Loop quantum gravity, causal dynamical triangulations and causal sets keep causal ordering or a background condition as primitive. N4 treats universality as an empirical limiting case; T22 supplies one topological conservation law independently of Noether. Chaitin supplies the metric on which all of these can be compared without an external judge: comprehensibility equals compressibility.

Part X — Epistemic Statuses

Every statement in this document carries exactly one of five statuses:

- **Assumed** — the ground (the trinity), the axioms N1–N5, and the hypotheses named in a theorem.
- **Proved** — T12, T16, T17, T18: pure arithmetic, no hypothesis beyond the integers; T1, T2 and T7.1: arithmetic of balanced ternary, given the ground.
- **Conditionally proved** — all other theorems, valid under their stated hypotheses (T15: equal blocks; Part VII: embedding in space; T21: unit-quaternion orientation states; T22: field maps $S^3 \to S^2$; T23: the stated energy density; T24: a free random path).
- **Corresponded** — the readings of II.3.
- **Open** — IX.1–IX.5.

Predictions and identifications are not part of this document; they are in the companion document. By T12.2 the status system itself gains a small foundation: readout at finite depth is well-defined without external convention. The rule of this document — say at which depth you are reading — is backed by arithmetic.

Appendix A — Record of the Checks

Every theorem and every number in this document was checked with code written for the purpose, on ranges fixed before running. Rows marked † were checked on 17 October 2026 for the Eighth Edition and are taken over without re-running; all other rows were run on 27 October 2026. Exact arithmetic (integers, fractions, sympy) was used wherever the values are rational.

Item	What was computed	Range or method	Result
T1	smallest cyclic group with $x \neq -x$	$\mathbb{Z}_1$ to $\mathbb{Z}_5$	$\mathbb{Z}_3$
III.1–III.2	r/ln r; facts (a)–(d); digit sets of radix ±3	all 3-digit sets in [−4, 4]	agrees; only {T, 0, 1} has both properties
T2–T6	windows; trit cube; closure probability	exhaustive; random laws	9 of 27; 3 + 6; 1 + 0 + 6 + 2; formula agrees
T7	reduction to standings, meetings, rings	all 265,720 closed words of length 1–12; all open words	all closed words reduce; no open word does
T7	shortest closures with nonzero steps	lengths 1–6	2 for w = 0, 3 for |w| = 1
T8	everywhere-closed words; first window	n = 3–9	9 for every n; 3^{n-1}
T9	rings closing everywhere	n = 3–13 labelled; n = 3, 6, 9 classes	9 or 3; 5 classes
T10	rewrite dynamics	n = 3–10, exhaustive	fixed points as stated; 24 rings at n = 4 and n = 8
T11	seam rule	all 81 ordered pairs	9 / 36 / 36
T12	existence, uniqueness, capacity	all 177,146 nonzero strings of length ≤ 11; Euler's product to K = 8	agrees
T12.3	digit count $\lfloor \log_3(2n-1) \rfloor + 1$	$n < 2 \times 10^5$, integer arithmetic	agrees
T13	necklace counts	n = 1–6	3, 6, 11, 24, 51, 130
T14	digit map in Z[ω]; norm descent	all a + bω with |a|, |b| ≤ 100; descent for |a|, |b| ≤ 60	ends at 0 or ±(1 + ω); 27.3 % and 41.0 %; no exception to descent
T15	equal-block condition	counterexample	4 ≠ 6

Item	What was computed	Range or method	Result
T16–T18	records of $\|\|k \cdot \log_2 3\|\|$; comma; residues	$k \le 10^6$; 60-digit decimals	list of T17; 1.013643; table of T18
T19, T20	Fox three-colourings; bracket; Jones polynomial	all Gauss words to 3 crossings; state sum over 8 states	unknot 3, trefoil 9; $\langle K \rangle$ and V as stated; $V(e^{i\pi/3}) = i\sqrt{3}$
T21†	quaternion double cover	10^4 random unit quaternions	deviation $\le 2 \times 10^{-15}$
T22†	Hopf invariant as linking number	maps $[z_1^p : z_2^q]$, coprime p, q ≤ 3	equals pq
T23†	constitutive identity	symbolic; 735 exact-rational cases	agrees
T24	Watson integrals; cumulants; finite walks	50 digits; symbolic; exact distribution of V for $L = 10^4$ in d = 3	P_3, P_4, P_5 as printed; -3.9612 for $L = 10^4$

Appendix B — The Origin of the Chain

The steps of Part 0 appeared in this order in the author's published texts on constable.blog.

Step	First appearance	Text
Void → distinction → −1, 0, +1 with sum zero	2008–2009	"About the Void" (1-12-2008); "About the Logic of Creation" (12-10-2009); "About Dark Matter" (27-1-2009)
Closure: "Closed in Itself"	2008–2009	"Why Numbers are Waves" (3-9-2008); "About the Logic of Creation"
The same structure at every level	2008	"About the Trinity" (29-12-2008); ×3 per depth as Δ → Δ×Δ in "About Number and Magnitude" (9-1-2012)
Two ladders, ×2 and ×3	2011	"About the Sum of Things" (22-9-2011); "About the Vedic Square" (19-12-2011)

Step	First appearance	Text
Balanced ternary	2022	“About Antifragile Design” (19-11-2022); radix economy in “De Verborgen Boodschap van de Sri-Yantra” (3-5-2025)
Mark, re-entry, eigenform O(A) = A	January 2026	“Louis H. Kauffman: A Complete Vision of Reality” (13-1-2026); “Exploring Knot Theory: Kauffman’s Vision and Applications” (13-1-2026)
The self-rewriting net	February 2026	“Het universum als zichzelf herschrijvend topologisch netwerk” (17-2-2026)
Why three and not two	May 2026	“Licht voordat licht gemaakt werd” (10-5-2026)
One strand	July 2026	“The Strand That Knots Itself” (9-7-2026)
Window rule mod 3; closure depths	September–October 2026	“Closure Is Zero Force” (PDF, 23-9-2026); near-closures in “The Comma in the Net” (2-9-2026); the denominators 1, 2, 5, 12, 41, 53, 306 in “The Climate on the Ladder of Three” (PDF, 3-10-2026)
ɣ as a state function	October 2026	“The Vacuum.Net Theory: A Formal Statement” (16-10-2026); “What the Path Gives” (PDF, 16-10-2026)
The strand as a map; direction gives three values; elementary closures; seams; the two ladders from meetings and rings	October 2026	“Curves in the Vacuum.Net” (PDF, 24-10-2026)
The address of a closed path as a rotation class; the trit in Z[ω]	October 2026	“The Positron and the Cogwheel” (26-10-2026); “Round Trips: Periods, Complex Multiplication and the Vacuum.Net” (27-10-2026)

All texts are at https://constable.blog under their dates.

Annotated References

Leibniz, G.W. (1686), *Discours de métaphysique*, §5–6. The design principle: maximal richness from minimal means. The remark in III.1.

Chaitin, G. (2005), *Meta Math! The Quest for Omega*; (2006), “The Limits of Reason”, *Scientific American* 294(3). A theory is a program shorter than its data. The comparison metric of III.1 and IX.6.

Spencer-Brown, G. (1969), *Laws of Form*, Allen & Unwin. The mark of distinction, the laws of calling and crossing, and re-entry. The ground of Part II.

Kauffman, L.H. (1987), “State models and the Jones polynomial”, *Topology* 26(3), 395–407. The bracket and its state sum, behind T20.

Kauffman, L.H. (2005), “EigenForm”, *Kybernetes* 34(1/2), 129–150. The eigenform O(A) = A; the formal side of N2.

Stifel, M. (1544), *Arithmetica Integra*, p. 38. An early appearance of balanced ternary.

Knuth, D. (1997), *The Art of Computer Programming*, Vol. 2, pp. 195–213. Balanced-ternary arithmetic; truncation–rounding equivalence.

Andrews, G.E. (2007), “Euler’s *De Partitio numerorum*”, *Bull. AMS* 44(4), 561–573. Euler’s use of the unique-representation theorem behind T12.

Hayes, B. (2001), “Third Base”, *American Scientist* 89(6). Survey of ternary systems and radix economy.

Krinitsky, N.A., Mironov, G.A. & Frolov, G.D. (1963), “Programmno-upravlyaemaya mashina Setun’”, in *Programmirovanie*, ed. M.R. Shura-Bura, Fizmatgiz, Moscow. The balanced-ternary computer of Brusentsov and Sobolev.

Ma, S. et al. (2024), “The era of 1-bit LLMs: all large language models are in 1.58 bits”, arXiv:2402.17764. Neural networks on the alphabet {−1, 0, 1}.

Hardy, G.H. and Wright, E.M. (2008), *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press. Kronecker’s theorem (Theorem 439), behind T16; continued fractions.

Khinchin, A.Ya. (1964), *Continued Fractions*, University of Chicago Press. Best approximations and Lagrange’s theorem, behind T17. Chapter II, §6.

Reidemeister, K. (1927), “Elementare Begründung der Knotentheorie”, *Abh. Math. Sem. Univ. Hamburg* 5, 24–32. The three moves.

Fox, R.H. and Crowell, R.H. (1963), *Introduction to Knot Theory*, Ginn; Springer GTM 57 (1977). Three-colouring, the rule of T4 and T19.

Jones, V.F.R. (1985), “A polynomial invariant for knots via von Neumann algebras”, *Bull. AMS* 12(1), 103–111. The polynomial that T20 computes.

Przytycki, J.H. (1998), “3-coloring and other elementary invariants of knots”, *Banach Center Publications* 42, 275–295. Three-colourings and the Jones polynomial.

Lickorish, W.B.R. and Millett, K.C. (1986), “Some evaluations of link polynomials”, *Comment. Math. Helv.* 61, 349–359. The Jones polynomial at roots of unity.

Hopf, H. (1931), “Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche”, *Math. Ann.* 104, 637–665. The invariant behind T22.

Pólya, G. (1921), “Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz”, *Math. Ann.* 84, 149–160. Return of the random walk; the basis of T24.

Watson, G.N. (1939), “Three triple integrals”, *Quart. J. Math. Oxford* 10, 266–276. $P_3 = 0.340537$ in T24.

Montroll, E.W. (1956), “Random walks in multidimensional spaces, especially on periodic lattices”, *J. SIAM* 4(4), 241–260. Return probabilities in four and five dimensions; OEIS A086232 and A086233.

Ireland, K. and Rosen, M. (1990), *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer GTM 84, Chapter 9. The Eisenstein integers $\mathbb{Z}[\omega]$ and the prime $1 - \omega$ above 3, behind T14.

Streng, M. (2014), “Computing Igusa class polynomials”, *Mathematics of Computation* 83, 275–309; and the NWO project ReSCue (Leiden, 2026). Curves with complex multiplication by $\mathbb{Z}[\omega]$ as the setting in which T14 was found.

Rosen, R. (1991), *Life Itself*, Columbia University Press. Closure under efficient causation; the form of IX.2.